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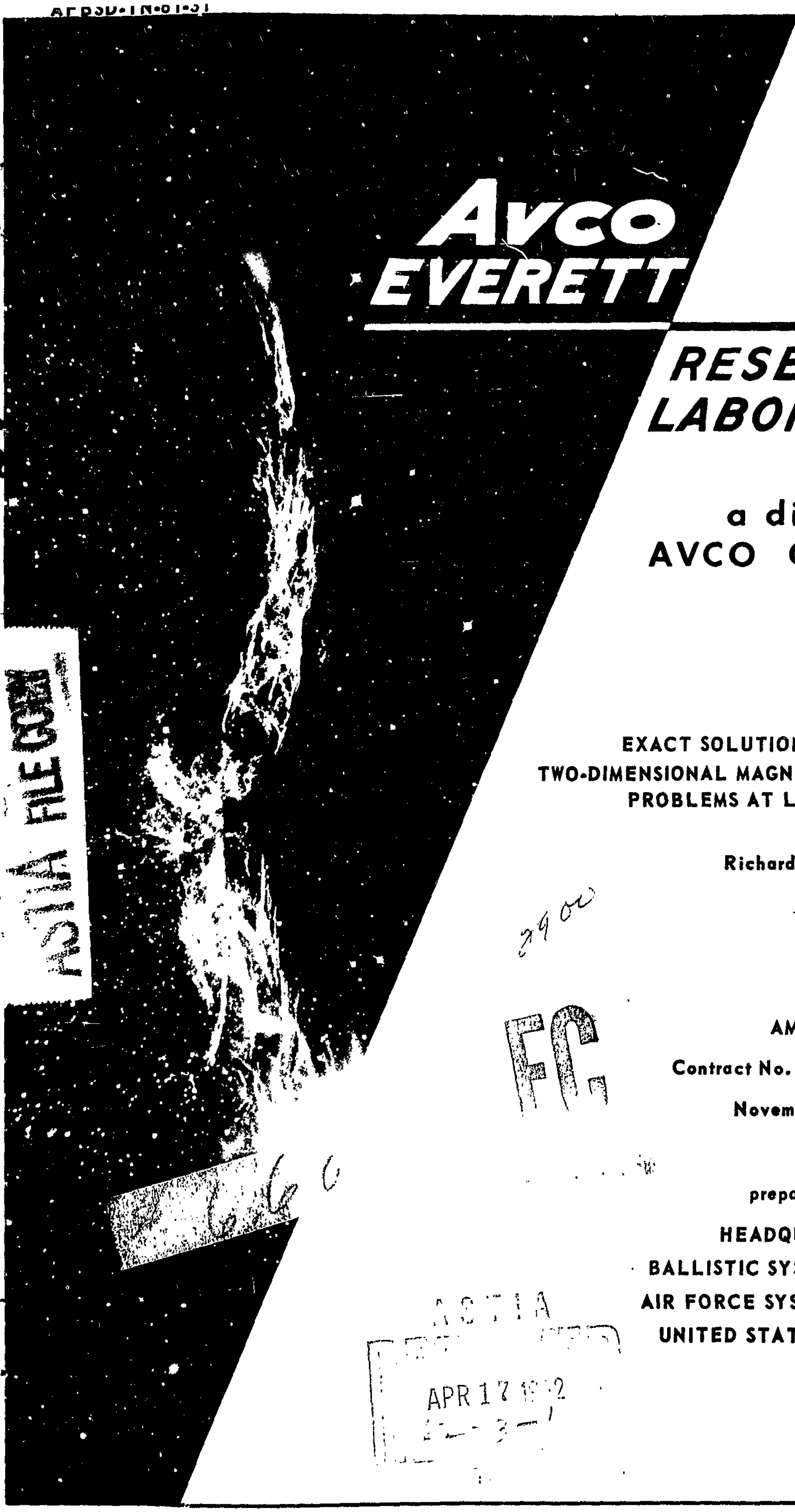

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EXACT SOLUTIONS TO A CLASS OF TWO-DIMENSIONAL MAGNETOHYDRODYNAMIC FLOW PROBLEMS AT LOW CONDUCTIVITY

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HEADQUARTERS
BALLISTIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
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AVCO-EVERETT RESEARCH LABORATORY
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Los Angeles 45, California

ABSTRACT

Exact solutions are presented to a number of small perturbation magnetohydrodynamic flow problems. The conditions under which the solutions are obtained are as follows:

1. The flow is two-dimensional, and is only slightly perturbed from a uniform flow.
2. The magnetic field vector is also two-dimensional and lies in the plane of the flow.
3. The distortion of the applied field by the induced currents is neglected.
4. Physical boundaries on the flow are one or two infinite plates parallel to the flow direction.
5. The conductivity of the fluid is a scalar quantity, but may vary with position.

With these assumptions, the perturbations to the flow are calculated for various magnetic fields (chiefly those due to a current flowing in a single wire, and a linear dipole) for incompressible, subsonic and supersonic free stream speeds. Calculations of the pressure on the walls and other quantities are presented for illustrative examples, including cases in which the conductivity is not uniform throughout the flow.

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LIST OF SYMBOLS

Dimensional quantities may be taken to be expressed in the rationalized MKS system of units.

Dimensional	Non-Dimensional	
$\underset{\sim}{j}^*$	$\underset{\sim}{j}$	current vector
j^*_z		z-component of the current vector
σ^*	σ', σ	conductivity
$\underset{\sim}{v}^*$		velocity vector
u^*, v^*	u', v'; u, v	velocity components
U		free stream velocity
$\underset{\sim}{B}^*$		magnetic field vector
B^*_x, B^*_y	b_x, b_y	components of the magnetic field vector
B_o		reference magnetic field strength
x^*, y^*	x, y	rectangular Cartesian coordinates
y_o		reference length
ρ^*	ρ', ρ	mass density
ρ_o		reference mass density
p^*	p', p	pressure
p_o		reference pressure
s^*	s', s	entropy
T^*		temperature

Dimensional	Non-Dimensional	
R		gas constant
	γ	ratio of specific heat
	S	interaction parameter
	M	Mach number
	ξ, n, t, ζ	dummy variables
	z	complex variable
	w	complex function of complex variable
	β	$\sqrt{\lvert M^2 - 1\rvert}$
	H, X	lengths
	λ, μ	complex variables
	Ψ	special function
	ψ	stream function
μ_o		permeability of free space
I		current
	$F(x)$	complex function of real variable

$\mathcal{C}_1$ $\mathcal{C}_2$; contours.

Subscripts P and c refer to particular and complementary solutions.

1. Introduction

A number of interesting steady flow problems in inviscid magneto-hydrodynamics may be treated approximately by neglecting entirely the magnetic Reynolds number, and assuming further that the interaction parameter (as defined, say, in Ref. 1) is small. In this paper solutions are given to a number of these problems under the following additional restrictions: first, the flow is two-dimensional; second, the magnetic field lies in the plane of the flow; third, the flow is only slightly perturbed from a uniform flow, the perturbation being of the first order in the interaction parameter.

Possibly the simplest flow of this type is that considered by Sherman;[2] the flow, which in this case is taken to be incompressible, is through an infinite channel of arbitrary height, while the magnetic field is due to a current flowing in a single wire located outside the channel and running perpendicular to the flow direction. This problem was treated numerically; the resulting pressure gradient on the surface of the channel was used in a subsequent paper[3] to investigate the boundary layer over one wall of the channel.

A similar problem in the sense of being a small perturbation on a uniform flow was discussed by Kemp and Petschek,[4] but the geometrical configuration treated in that paper was somewhat different.

It will be shown in this paper that the solution to the channel problem may be obtained analytically; the case in which the channel height is infinite has a simple closed-form solution. In addition, a method will be given for solving similar channel flow problems for any two-dimensional magnetic field arising from an arrangement of pole pieces or current carrying wires outside the channel. Finally, solutions are given to some linearized subsonic and supersonic compressible flow problems with the same geometry, and some incompressible problems in which the conductivity is not uniform.

2. General Analysis

In the absence of an applied electric field the current is given by the expression

$$\underset{\sim}{j}^* = \sigma^* \underset{\sim}{v}^* \times \underset{\sim}{B}^* \tag{2.1}$$

Taking Cartesian coordinates (x^*, y^*) in the plane of the flow, with the x-axis parallel to the undisturbed flow, it is found that the current is entirely in the z-direction and is given by

$$j_z^* = \sigma^*(u^* B_y^* - v^* B_x^*) \tag{2.2}$$

The body force on the fluid is given by

$$\underset{\sim}{j}^* \times \underset{\sim}{B}^* = (-j_z^* B_y^*, j_z^* B_x^*, 0) \tag{2.3}$$

and the equations of motion become

$$\rho^*\left(u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right)+\frac{\partial p^*}{\partial x^*}+j_z^*B_y^*=0 \tag{2.4}$$

$$\rho^*\left(u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}\right)+\frac{\partial p^*}{\partial y^*}-j_z^*B_x^*=0 \tag{2.5}$$

In addition, the equation of continuity is

$$\frac{\partial}{\partial x^*}(\rho^* u^*)+\frac{\partial}{\partial y^*}(\rho^* v^*)=0 \tag{2.6}$$

and the energy equation is

$$u^*\frac{\partial s^*}{\partial x^*}+v^*\frac{\partial s^*}{\partial y^*}=\frac{{j^*}^2}{\sigma^*\rho^*T^*} \tag{2.7}$$

Finally, we assume that the fluid is a perfect gas with constant ratio of specific heats; so that

$$p^*=\rho^*RT^* \tag{2.8}$$

$$ds^*=\frac{R}{\gamma-1}\left\{\frac{dp^*}{p^*}-\gamma\frac{d\rho^*}{\rho^*}\right\} \tag{2.9}$$

The following non-dimensional quantities will now be defined:

$$x=\frac{x^*}{y_0}\ ,\quad y=\frac{y^*}{y_0}\ ,\quad j=\frac{j^*}{\sigma^*UB_0}\ ,$$

$$b_x = \frac{B_x^*}{B_o}, \quad b_y = \frac{B_y^*}{B_o}, \quad u' = \frac{U^*}{U}$$

$$v' = \frac{v^*}{U}, \quad \rho' = \frac{\rho^*}{\rho_o}, \quad s' = \frac{s^*}{R},$$

$$p' = \frac{p^*}{\rho_o U^2}$$

(2.10)

In terms of these variables, Eqs. (2.4) - (2.9) become:

$$\rho'\left(u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y}\right) + \frac{\partial p'}{\partial x} + S j_z b_y = 0$$

(2.11)

$$\rho'\left(u'\frac{\partial v'}{\partial x} + v'\frac{\partial v'}{\partial y}\right) + \frac{\partial p'}{\partial y} - S j_z b_x = 0$$

(2.12)

$$\frac{\partial}{\partial x}(\rho' u') + \frac{\partial}{\partial y}(\rho' v') = 0$$

(2.13)

$$u'\frac{\partial s'}{\partial x} + v'\frac{\partial s'}{\partial y} = S\frac{j^2}{p'}$$

(2.14)

$$ds' = \frac{1}{\gamma - 1}\left\{\frac{dp'}{p'} - \gamma\frac{d\rho'}{\rho'}\right\} \qquad (2.15)$$

Where S (the interaction parameter) is defined by

$$S = \frac{\sigma^* B_0^2 y_0}{\rho U} \qquad (2.16)$$

The magnetic field is assumed to be unaffected by the currents flowing in the gas (low magnetic Reynolds number); this assumption will always break down far from the sources of the field, but will be satisfactory for the purposes of this paper. In these circumstances, Maxwell's equations will require the magnetic field to satisfy

$$\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} = 0 \qquad (2.17)$$

$$\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y} = 0 \qquad (2.18)$$

Subject to these conditions, however, the magnetic field will be a given quantity in any particular problem.

The flows to be considered in this paper will be only slightly perturbed from a uniform parallel flow; perturbation quantities are defined as follows:

$$u' = 1 + Su \qquad (2.19)$$

$$v' = Sv \qquad (2.20)$$

$$\rho' = 1 + S\rho \tag{2.21}$$

$$s' = Ss \tag{2.22}$$

$$p' = \frac{1}{\gamma M^2} + Sp \tag{2.23}$$

where $M^2 = \frac{\rho_o U^2}{\gamma\, p_o}$ is the free stream Mach number. Substituting Eqs. (2.19) (2.23) in Eqs. (2.11) - (2.15) and taking only terms of order S then gives:

$$\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = -b_y^2 \tag{2.24}$$

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} = b_x b_y \tag{2.25}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \rho}{\partial x} = 0 \tag{2.26}$$

$$\frac{\partial s}{\partial x} = \gamma M^2 b_y^2 \tag{2.27}$$

$$ds = \frac{1}{\gamma - 1}\left\{\gamma M^2 dp - \gamma d\rho\right\} \tag{2.28}$$

The last equation may be integrated at once to give

$$s = \frac{\gamma M^2}{\gamma - 1} p - \frac{\gamma}{\gamma - 1} \rho \tag{2.29}$$

Equations (2.24) - (2.27) and (2.29) present the problem under consideration in a form suitable for analysis, but before considering any particular problem, some general conclusions may be drawn. First, Eq. (2.27) may be integrated at once to give

$$s(x,y) = \gamma M^2 \int_{-\infty}^{x} b_y^{\,2}(\xi, y)\, d\xi \tag{2.30}$$

Second, the linearized momentum equations, (2.24) and (2.25), now describe a flow moving under a specified non-conservative body force; this force is calculated directly from the given field quantities. Third, the x-momentum Eq. (2.24) may be integrated to give

$$u + p = -\int_{-\infty}^{x} b_y^{\,2}(\xi, y)\, d\xi \tag{2.31}$$

Next, the terms in u and ρ may be eliminated from Eq. (2.26) by using Eqs. (2.24), (2.27) and (2.29) to give:

$$\frac{\partial v}{\partial y} + (M^2 - 1)\frac{\partial p}{\partial x} = \left[1 + (\gamma - 1) M^2\right] b_y^{\,2} \tag{2.32}$$

Now, if Eqs. (2.25) and (2.32) can be solved for v and p, u may be found from Eq. (2.31), and ρ from Eqs. (2.29) and (2.30), thus completing the solution of the problem.

3. Incompressible Problems

In the incompressible case Eqs. (2.25) and (2.32) may be written

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} = b_x b_y \tag{3.1}$$

$$\frac{\partial v}{\partial y} - \frac{\partial p}{\partial x} = b_y^2 \tag{3.2}$$

which equation together with Eq. (2.31):

$$u = -p - \int_{-\infty}^{x} b_y^2(\xi, y)\, dx \tag{3.3}$$

define the problem.

It is a remarkable fact that a particular solution of Eqs. (3.1) and (3.2) may be found for an arbitrary magnetic field. This solution is:

$$v_p = \frac{1}{2} b_y \int_{-\infty}^{x} b_x(\xi, y)\, d\xi \tag{3.4}$$

$$p_p = -\frac{1}{2} b_x \int_{-\infty}^{x} b_x(\xi, y)\, d\xi + \frac{1}{2} \int_{-\infty}^{x} \left\{ b_x^2(\xi, y) - b_y^2(\xi, y) \right\} d\xi \tag{3.5}$$

The integrals in these expressions will be convergent in all normal cases. The complete solution to Eqs. (3.1) and (3.2) can now be written

$$\left.\begin{aligned} v &= v_p + v_c \\ p &= p_p + p_c \end{aligned}\right\} \tag{3.6}$$

where v_c and p_c satisfy

$$\begin{aligned} \frac{\partial v_c}{\partial x} + \frac{\partial p_c}{\partial y} &= 0 \\ \frac{\partial v_c}{\partial y} - \frac{\partial p_c}{\partial x} &= 0 \end{aligned} \tag{3.7}$$

Clearly, Eq. (3.7) may be satisfied by writing

$$P_c + i v_c = w(x+iy) = w(z) \tag{3.8}$$

where w (z) is an analytic function of the complex variable z.

The boundary condition appropriate to any particular problem will require the normal component of the velocity to vanish at a solid surface; in this paper we will consider only infinite channels defined by y = 0 and y = H, say, together with the limiting case $H \rightarrow \infty$. On these surfaces the boundary condition then reduces to

$$\Im\, w + v_p = 0 \tag{3.9}$$

Single Wire Case

Turning now to particular cases, the simplest magnetic field is that due to a current I in an infinite wire through the point (0, -1) with the lower wall of the channel at y = 0. The physical distance of the wire from the wall is y_o, and B_o may be taken to be the field at the origin, $\mu_o I/2\pi y_o$. Then

$$b_x = \frac{(y+1)}{x^2+(y+1)^2} \qquad b_y = \frac{-x}{x^2+(y+1)^2} \tag{3.10}$$

and

$$v_p = \frac{-\frac{1}{2}\, x \tan^{-1} \frac{x}{y+1}}{x^2+(y+1)^2} \tag{3.11}$$

$$p_p = \frac{\frac{1}{2}\left[x-(y+1)\tan^{-1}\frac{x}{y+1}\right]}{x^2+(y+1)^2} \tag{3.12}$$

In deriving these expressions from Eqs. (3.4) and (3.5) all the integrals were taken from the lower limits $\zeta = 0$ instead of $\zeta = -\infty$. It may be verified that in this particular case Eqs. (3.11) and (3.12) are still particular solutions to Eqs. (3.1) and (3.2). A slight simplification of the algebra results from this unimportant modification.

Considering first the case of infinite channel height, w (z) must be analytic for $\Im z > 0$, and satisfy:

$$\mathscr{I}\, w(z) = \frac{\frac{1}{2} x \tan^{-1} x}{x^2+1} \quad \text{on } y=0$$

(3.13)

The solution may be obtained by standard methods and is easily shown to be

$$w(z) = \frac{-z \ln \frac{1-iz}{2}}{2(z^2+1)}$$

(3.14)

The complete flow field is now easily established, but it will be sufficient here to derive the more important features of this flow. The pressure on the wall is given by

$$p(x,0) = \frac{x - \tan^{-1} x - \frac{1}{2} x \ln \frac{1+x^2}{4}}{2(x^2+1)}$$

(3.15)

In addition, we find

$$v(0,y) = \frac{y \ln \frac{1+y}{2}}{2(y^2-1)}$$

(3.16)

and

$$2u(0,y) = u(\infty\ y) = -\frac{\pi/2}{y+1}$$

(3.17)

This profile is, of course, rotational, and represents the amount of vorticity produced by the non-conservative force field. Finally, the pressure gradient on the wall is found from Eq. (3.15) to be

$$\left.\frac{\partial p}{\partial x}\right|_{y=0} = \frac{-x(x-\tan^{-1}x) + \frac{1}{4}(x^2-1)\ln \frac{x^2+1}{4}}{(x^2+1)^2}$$

(3.18)

At the origin this gives $\frac{\partial p}{\partial x}(0,\ 0) = \ln\sqrt{2}$. Equations (3.15) and (3.18) are shown in Figs. 1 and 2, together with material to be described later.

Single Wire and Channel

Next, consider an infinitely long channel with walls at y = 0 and y = H, the magnetic field being the same as in the previous case. v_p and p_p are still given by Eqs. (3.11) and (3.12), but, in addition to Eq. (3.13), w (z) must satisfy

$$\mathscr{I}\, w = \frac{\frac{1}{2} x \tan^{-1} \frac{x}{H+1}}{x^2 + (H+1)^2} \quad \text{on } y = H \tag{3.19}$$

Defining

$$\lambda = \frac{1 - iz}{2H} \qquad \mu = \frac{1 + iz}{2H} \tag{3.20}$$

w (z) may be shown to be given by:

$$w(z) = \frac{\pi \ln (H+1)}{8H} \left[\coth i\pi\mu - \coth i\pi\lambda - 2\right] - \frac{z \ln \lambda}{2 (z^2+1)}$$

$$- \frac{1}{8Hi} \sum_{n=1}^{\infty} \left[\frac{\ln (n+\mu)}{(n-\lambda)(n-1-\lambda)} - \frac{\ln (n+\lambda)}{(n-\mu)(n-1-\mu)} \right] \tag{3.21}$$

Interesting deductions that can be derived from Eq. (3.21) without summing the series are:

$$2u(0,y) = u(\infty,y) = \frac{\pi}{2} \left[\frac{\ln (H+1)}{H} - \frac{1}{y+1} \right] \tag{3.22}$$

$$2p(0,y) = p(\infty,y) = \frac{-\pi \ln (H+1)}{2H} \tag{3.23}$$

This expression for the pressure drop along the channel is plotted in Fig. 3. For finite channel flows of this type, the drag is just the product of the channel height and the pressure drop. However, it is not necessary to calculate the flow field to determine the drag; to this order it is only necessary to integrate the magnetic force through the field. When $H \rightarrow \infty$ the drag is logarithmically infinite, a result which is due to the slow way in which the field decreases at large distances from the wire. The pressure on the lower wall (y = 0) is given by

$$P(x,0) = \frac{-\pi \ln(H+1)}{4H}\left[\frac{\operatorname{sh} x\,\pi/H}{\operatorname{ch}\frac{x\pi}{H} - \cos\pi/H} + 1\right] + \frac{\frac{1}{2}(x - \tan^{-1}x) - \frac{1}{4}x \ln\frac{x^2+1}{4}}{x^2+1}$$

$$-\frac{1}{4H}\sum_{n=1}^{\infty}\frac{\left\{(n-\frac{1}{2H})(n-1-\frac{1}{2H}) - (\frac{x}{2H})^2\right\}\tan^{-1}\frac{x}{2nH+1} - \frac{x}{2H}(2n-1-\frac{1}{H})\frac{1}{2}\ln\frac{x^2+(2nH+1)^2}{4}}{\left[(n-\frac{1}{2H})^2 + (\frac{x}{2H})^2\right]\left[(n-1-\frac{1}{2H})^2 + (\frac{x}{2H})^2\right]} \tag{3.24}$$

This expression is shown for various values of H in Fig. 1, and its derivative in Fig. 2. It can be shown that for any finite H the pressure tends to its asymptotic value from above, while for infinite H the reverse is true--an interesting case of non-uniform convergence. The curves for H = 0 correspond to the assumption v = 0 (from continuity) and come from integrating Eq. (3.3) on this basis.

The results obtained for the pressure gradient appear to agree quite closely with those of Sherman;[3] the actual pressures differ by a nearly constant amount; this is probably due to the fact that the numerical integration was commenced by setting p = 0 at some finite upstream distance and is of no importance. The pressure gradient is unfavorable for a boundary layer on the wall over a distance of the order of unity near the wire. This unfavorable region shrinks with the channel height.

Two Wires

As an example of a slightly more complicated magnetic field, as well as a more realistic case in the sense that the net current giving rise to the magnetic field is zero, consider two wires located at the points (X, -1), (-X, -1) carrying equal currents I in opposite directions. The distance of either wire from the wall y = 0 is y_o and the distance between the wires is $2Xy_o$. B_o will again be taken to be $-\mu_o I/2\pi y_o$, so that the non-dimensional field components are

$$b_x = \frac{Y+1}{(x+X)^2+(Y+1)^2} - \frac{(Y+1)}{(x-X)^2+(Y+1)^2}, \quad b_Y = \frac{-(x+X)}{(x+X)^2+(Y+1)^2} + \frac{(x-X)}{(x-X)^2+(Y+1)^2} \tag{3.25}$$

The particular solution in this case is:

$$v_p = \frac{1}{2}\left(\tan^{-1}\frac{x+X}{Y+1} - \tan^{-1}\frac{x-X}{Y+1}\right)\left[\frac{-(x+X)}{(x+X)^2+(Y+1)^2} + \frac{(x-X)}{(x-X)^2+(Y+1)^2}\right] \tag{3.26}$$

$$P_p = \frac{2xX(Y+1)}{[(x+X)^2+(Y+1)^2][(x-X)^2+(Y+1)^2]}\left[\tan^{-1}\frac{x+X}{Y+1} - \tan^{-1}\frac{x-X}{Y+1}\right] + \frac{1}{2}\left[\frac{x+X}{(x+X)^2+(Y+1)^2} + \frac{(x-X)}{(x-X)^2+(Y+1)^2}\right] + \frac{1}{4X}\ln\frac{(x-X)^2+(Y+1)^2}{(x+X)^2+(Y+1)^2} \tag{3.27}$$

so that for the single wall case w must satisfy

$$\vartheta\, w = \frac{1}{2}\left(\tan^{-1}(x+X) - \tan^{-1}(x-X)\right)\left[\frac{x+X}{1+(x+X)^2} - \frac{x-X}{1+(x-X)^2}\right] \quad \text{on } Y=0 \tag{3.28}$$

The solution to this problem is

$$w = -\frac{\frac{1}{2}(z+X)\ln\frac{1-i(z+X)}{2}}{(z+X)^2+1} - \frac{\frac{1}{2}(z-X)\ln\frac{1-i(z-X)}{2}}{(z-X)^2+1} + \frac{1}{2}\,\frac{(z+X)\ln\frac{z-X+i}{2(i-X)}}{(z+X)^2+1} + \frac{1}{2}\,\frac{(z-X)\ln\frac{z+X+i}{2(X+i)}}{(z-X)^2+1} - \frac{\frac{1}{2}i\tan^{-1}X}{z-X+i} + \frac{\frac{1}{2}i\tan^{-1}X}{z+X+i} \tag{3.29}$$

Hence

$$2u(0,Y) = u(\infty,Y) = \frac{-\pi X^2}{(Y+1)\left[X^2+(Y+1)^2\right]} \tag{3.30}$$

$$p(0,Y) = 0 \tag{3.31}$$

$$p(x,0) = \frac{2xX}{\left[(x-X)^2+1\right]\left[(x+X)^2+1\right]}\left[\tan^{-1}(x+X) - \tan^{-1}(x-X) - \tan^{-1}X\right] + \frac{\frac{1}{2}(x+X)}{(x+X)^2+1} + \frac{\frac{1}{2}(x-X)}{(x-X)^2+1}$$

$$- \frac{\frac{1}{4}(x+X)\ln\frac{(1+X^2)\{1+(x+X)^2\}}{1+(x-X)^2}}{(x+X)^2+1} - \frac{\frac{1}{4}(x-X)\ln\frac{(1+X^2)\{1+(x-X)^2\}}{1+(x+X)^2}}{(x-X)^2+1} + \frac{1}{4X}\ln\frac{(x-X)^2+1}{(x+X)^2+1} \tag{3.32}$$

The drag in this case is finite; it can be found from integrating Eq. (3.30), but it can also be found from integrating the force in the x-direction through the whole flow field without calculating the actual velocity components. Either way, the result is $\frac{\pi}{2}\ln(1+X^2)$. The dimensional drag per unit length is then $\sigma U B_o^2 y_o^2 \cdot \frac{\pi}{2}\ln(1+X^2) = \sigma U \mu_o^2 I^2 \cdot \frac{1}{8\pi}\ln(1+X^2)$. The pressure on the wall Eq. (3.32) is shown in Fig. 4 for a number of values X; the pressure gradient is shown in Fig. 5.

For comparative purposes the product of the current in either wire times the distance between the wires has been kept constant. This has the effect of making the reference field equal to the y-component of the field at the origin and permits transition to the case of the linear dipole. For large X the two wires behave independently as may be seen by comparing with the previous case. However, as X gets smaller the character of the pressure distribution changes; the pressure gradient becomes unfavorable over a longer distance, but has smaller values. The curves for X = 0.1 are indistinguishable from those for the dipole case (X = 0) which will be presented next.

Linear Dipole

Particularly simple results may be found in the case of the linear dipole. If this is placed at the point (0, -1)(distant y_o from the wall) and the magnetic field at (0, 0) is taken to be B_o, the non-dimensional components of the magnetic field are:

$$b_x = \frac{-2X(Y+1)}{[X^2+(Y+1)^2]^2} \qquad b_y = \frac{X^2-(Y+1)^2}{[X^2+(Y+1)^2]^2}$$

(3.33)

This case may also be derived from the previous case by letting I become large and X become small in such a way that $2XI = 2\pi B_o y_o/\mu_o$. The particular solution is:

$$v_p = \frac{\frac{1}{2}(Y+1)[X^2-(Y+1)^2]}{[X^2+(Y+1)^2]^3}$$

(3.34)

$$P_p = \frac{\frac{1}{2}X[(Y+1)^2+\frac{1}{3}X^2]}{[X^2+(Y+1)^2]^3}$$

(3.35)

so that, for the infinite channel height case,

$$\mathcal{J}\,w = \frac{-\frac{1}{2}(X^2-1)}{(X^2+1)^3} \qquad \text{on } Y=0$$

(3.36)

Hence

$$w = -\tfrac{1}{8}(z^2+3iz-4)(z+i)^{-3}$$

(3.37)

The complete solution is then:

$$u = \frac{-\frac{1}{4}\left(\frac{\pi}{2}+\tan^{-1}\frac{X}{Y+1}\right)}{(Y+1)^3} + \frac{\frac{1}{8}X[X^2(Y^2+2Y-1)+(Y+1)^2(Y^2+4Y-1)]}{(Y+1)^2[X^2+(Y+1)^2]^2}$$

(3.38)

$$v = \frac{\frac{1}{8}Y[X^2+(Y+1)(Y+3)]}{[X^2+(Y+1)^2]^2}$$

(3.39)

$$p = -\frac{\frac{1}{8}x\left[x^4 + 2x^2\left(y^2 + 3y + \frac{1}{3}\right) + (y+1)^2\left(y^2 + 4y + 5\right)\right]}{\left[x^2 + (y+1)^2\right]^3} \tag{3.40}$$

Also, the stream function defined by

$$\psi_x = -v \qquad \psi_y = u \tag{3.41}$$

is found to be

$$\psi = -\frac{1}{8}\,\frac{y(y+2)}{(y+1)^2}\left(\frac{\pi}{2} + \tan^{-1}\frac{x}{y+1}\right) - \frac{1}{8}\,\frac{xy}{(y+1)\left\{x^2 + (y+1)^2\right\}} \tag{3.42}$$

The pressure gradient on the wall is

$$\frac{\partial}{\partial x}\, p(x,0) = \frac{x^6 - 3x^4 + 23x^2 - 5}{8(x^2+1)^4} \tag{3.43}$$

Lastly, the streamline which is defined by $\psi = -y_o$ rises in passing the dipole, the amount of the rise being

$$\Delta y = \frac{S\pi}{8}\;\frac{y_o(y_o+2)}{(y_o+1)^2} \tag{3.44}$$

The pressure and the pressure gradient are shown in Figs. 4 and 5.

Linear Dipole and Channel

If a finite channel is introduced having walls at y = 0 and y = H, the conditions on v_c are Eq. (3.36) and

$$\vartheta w = -\tfrac{1}{2}(H+1)\left[x^2 - (H+1)^2\right] \Big/ \left[x^2 + (H+1)^2\right]^3 \tag{3.45}$$

The complementary solution is then

$$w = -\frac{[z^2+3iz-4]}{8(z+1)^3} - \frac{i(H+2)}{16(H+1)^2}\left[\Psi(\lambda)-\Psi(\mu)-i\pi\right] - \frac{i}{32H(H+1)}\left[\Psi'(\mu)-\Psi'(\lambda)\right] \tag{3.46}$$

where $\lambda = \frac{1-iz}{2H}$, $\mu = \frac{1+iz}{2H}$, and Ψ and Ψ' are defined by

$$\begin{aligned} \Psi(z) &= \frac{d}{dz}\ln z! = \frac{d}{dz}\ln\Gamma(z+1) \\ \Psi'(z) &= \frac{d^2}{dz^2}\ln z! = \frac{d\Psi}{dz} \end{aligned} \tag{3.47}$$

as in Ref. 5. Thus:

$$2u(0,y) = u(\infty,y) = \frac{\pi}{8}\left[\frac{(H+2)}{(H+1)^2} - \frac{2}{(y+1)^3}\right] \tag{3.48}$$

$$2p(0,y) = p(\infty,y) = -\frac{\pi}{8}\,\frac{(H+2)}{(H+1)^2} \tag{3.49}$$

The last expression gives the pressure drop along the channel and is shown in Fig. 6. The pressure on the wall $y = 0$ is

$$\begin{aligned} P(x,0) = {} & \frac{-x(x^4+\frac{2}{3}x^2+6)}{8(x^2+1)^3} - \frac{\pi(H+2)}{16(H+1)^2} - \frac{x(H+2)}{16H(H+1)^2}\sum_{n=1}^{\infty}\frac{1}{(n+\frac{1}{2H})^2+(\frac{x}{2H})^2} \\ & - \frac{x}{16H^2(H+1)}\sum_{n=1}^{\infty}\frac{n+\frac{1}{2H}}{\left[(n+\frac{1}{2H})^2+(\frac{x}{2H})^2\right]^2} \end{aligned} \tag{3.50}$$

and is shown in Fig. 7. The effect of the channel in this case, as in the case of the single wire is again to reduce the region of unfavorable pressure gradient; eventually, as $H \to 0$ the pressure drops continuously along the channel.

4. Subsonic Problems

No solution of the generality of Eqs. (3.4) and (3.5) is known in the case of subsonic compressible flow. It is not sufficient, for the inhomogeneous problem, merely to scale the quantities according to the Prandtl-Glauert rule. This is because the existence of the particular solution Eqs. (3.4) and (3.5) depends crucially on the relations Eqs. (2.17) and (2.18) connecting the components of the magnetic field; these relations are modified in the scaling procedure leaving the problem in a new form that is apparently no simpler than the original one. However, particular solutions can nevertheless be found for simple field configurations; once such a particular solution is found, for some magnetic field, the homogeneous problem becomes a straightforward problem in complex variables whatever the nature of the boundaries.

Single Wire Case

First, take the magnetic field as due to a current in single wire, the components being given by Eq. (3.10). The equations for v and p are:

$$\frac{\partial v}{\partial y} - (1-M^2)\frac{\partial p}{\partial x} = \frac{[1+(\gamma-1)M^2]x^2}{[x^2+(y+1)^2]^2} \tag{4.1}$$

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} = \frac{-x(y+1)}{[x^2+(y+1)^2]^2} \tag{4.2}$$

By transforming to polar coordinates with origin at the wire, it can be seen that particular solutions of Eqs. (4.1) and (4.2) exist in which v and p each have the form of a function of the angle divided by the first power of the distance; on transforming these particular solutions back into Cartesians, one obtains:

$$v_p = \frac{-\frac{1}{2}[1+(\gamma-1)M^2]x\left(\tan^{-1}\frac{x}{y+1}\right)}{x^2+\beta^2(y+1)^2} + \frac{1}{2}\gamma(y+1)\left[\frac{1}{x^2+(y+1)^2} - \frac{\beta^2}{x^2+\beta^2(y+1)^2}\right] \tag{4.3}$$

$$p_p = \frac{-\frac{1}{2}[1+(\gamma-1)M^2](y+1)\left(\tan^{-1}\frac{x}{y+1}\right)}{x^2+\beta^2(y+1)^2} + \frac{1}{2}x\left[\frac{-(\gamma-1)}{x^2+(y+1)^2} + \frac{\gamma}{x^2+\beta^2(y+1)^2}\right]$$

where $\beta^2 = 1 - M^2$ for subsonic flow. (4.4)

The complementary solution v_c, p_c satisfies:

$$\frac{\partial v_c}{\partial y} - \beta^2 \frac{\partial p_c}{\partial x} = 0 \tag{4.5}$$

$$\frac{\partial v_c}{\partial x} + \frac{\partial p_c}{\partial y} = 0 \tag{4.6}$$

and these equations will be satisfied by an analytic function $w_c(z)$ defined by:

$$w_c(z = x + i\beta\, y) = \beta p_c + i v_c \tag{4.7}$$

Taking the case of a single wall at y = 0, the condition on v_c is:

$$v_c(x,0) = \frac{\frac{1}{2}\left[1+(\gamma-1)M^2\right] x \tan^{-1} x}{x^2+\beta^2} - \frac{1}{2}\gamma\left[\frac{1}{x^2+1} - \frac{\beta^2}{x^2+\beta^2}\right] \tag{4.8}$$

The solution of this problem is

$$w_c(z) = \frac{-\frac{1}{2}\left[1+(\gamma-1)M^2\right] z \ln \frac{1-iz}{1+\beta}}{z^2+\beta^2} + \frac{1}{2}\gamma\left[\frac{1}{z+i} - \frac{\beta}{z+i\beta}\right] \tag{4.9}$$

From this may be found all the remaining flow quantities. For example:

$$p(x,0) = \frac{-\frac{1}{2}\left[1+(\gamma-1)M^2\right]}{x^2+\beta^2}\left\{\frac{1}{2}\frac{x}{\beta}\ln\frac{1+x^2}{(1+\beta)^2} + \tan^{-1} x\right\} + \frac{\frac{1}{2}x\left\{\frac{\gamma}{\beta} - (\gamma-1)\right\}}{x^2+1} \tag{4.10}$$

Hence, from Eq. (2.31)

$$u(x,0)=\frac{\frac{1}{2}[1+(\gamma-1)M^2]}{x^2+\beta^2}\left\{\frac{1}{2}\frac{x}{\beta}\ln\frac{1+x^2}{(1+\beta)^2}+\tan^{-1}x\right\}+\frac{\frac{1}{2}\gamma x(1-\frac{1}{\beta})}{x^2+1}-\frac{1}{2}\left(\frac{\pi}{2}+\tan^{-1}x\right)$$

(4.11)

Furthermore, from Eq. (2.30),

$$s(x,0)=\frac{1}{2}\gamma M^2\left[\left(\frac{\pi}{2}+\tan^{-1}x\right)-\frac{x}{x^2+1}\right]$$

(4.12)

and from Eq. (2.29)

$$\rho(x,0)=\frac{-\frac{1}{2}[1+(\gamma-1)M^2]M^2}{x^2+\beta^2}\left[\frac{1}{2}\frac{x}{\beta}\ln\frac{1+x^2}{(1+\beta)^2}+\tan^{-1}x\right]+\frac{\frac{1}{2}x\gamma M^2/\beta}{x^2+1}$$

$$-\frac{1}{2}(\gamma-1)M^2\left(\frac{\pi}{2}+\tan^{-1}x\right)$$

(4.13)

These quantities are shown in Figs. 8, 9. 10 and 11 for various values of M, and for $\gamma = 1.4$ and $\gamma = 1.0$. Comment on these graphs will be reserved until after the discussion of the supersonic case which will be given in the next section.

The channel problem can also be solved by the method of images, but the solution will not be presented here.

Linear Dipole

The transformation into polar coordinates can also be used to find the particular solution to the subsonic problem when the magnetic field is due to a linear dipole, the components of the magnetic field being given by Eq. (3.33). The particular solution is:

$$p_p=\frac{[1+(\gamma-1)M^2]M^2[3x^2-\beta^2(y+1)^2](y+1)}{4[x^2+\beta^2(y+1)^2]^3}\left(\frac{\pi}{2}-\tan^{-1}\frac{x}{y+1}\right)$$

$$+\frac{x}{12[x^2+(y+1)^2]^3[x^2+\beta^2(y+1)^2]^3}\left[2(3+\beta^2)x^8+3(7-\beta^2-2\beta^4)x^6(y+1)^2\right.$$

$$+3(8-5\beta^2-7\beta^4)x^4(y+1)^4+(9-13\beta^2-24\beta^4)x^2(y+1)^6-3\beta^2(1+3\beta)^2(y+1)^8\Big]$$

$$+\frac{(\gamma-1)M^2x}{12[x^2+(y+1)^2]^3[x^2+\beta^2(y+1)^2]^3}\Big[4x^8+3(9-\beta)x^6(y+1)^2+3(8+\beta^2-\beta^4)x^4(y+1)^4$$

$$+(9-13\beta^2-4\beta^4)x^2(y+1)^6-3\beta^2(1+3\beta^2)(y+1)^8\Big]$$

(4.14)

$$v_p=\frac{x[x^2-3\beta^2(y+1)^2]}{[x^2+\beta^2(y+1)^2]^3}\left[1+(\gamma-1)M^2\right]\frac{M^2}{4}\left(\frac{\pi}{2}-\tan^{-1}\frac{x}{y+1}\right)$$

$$+\frac{(\gamma-1)M^2x^2(y+1)}{12[x^2+(y+1)^2]^3[x^2+\beta^2(y+1)^2]^3}\Big[9(1-\beta^2)x^6+(4-23\beta^2-5\beta^4)x^4(y+1)^2$$

$$+3(1-13\beta^2-4\beta^4)x^2(y+1)^4-3\beta^2(3+5\beta^2)(y+1)^6\Big]$$

$$\frac{x^2(y+1)}{12[x^2+(y+1)^2]^3[x^2+\beta^2(y+1)^2]^3}\Big[3(3+\beta^2)(1-2\beta^2)x^6+(4-21\beta^2$$

$$-21\beta^4+2\beta^6)x^4(y+1)^2+(3-39\beta^2-6\beta^4+6\beta^6)x^2(y+1)^4+3\beta^2(-3-5\beta^2+4\beta^4)(y+1)^6\Big]$$

(4.15)

The homogeneous problem can again be solved by a modified complex variable as defined in Eq. (4.7). Taking the case of a single wall at $y = 0$, the condition on v_c is:

$$v_c(x,0)=-\frac{x(x^2-3\beta^2)}{(x^2+\beta^2)^3}\left[1+(\gamma-1)M^2\right]\frac{M^2}{4}\left(\frac{\pi}{2}-\tan^{-1}x\right)-$$

$$-\frac{(\gamma-1)M^2x^2}{12(x^2+1)^3(x^2+\beta^2)^3}\Big[9(1-\beta^2)x^6+(4-23\beta^2-5\beta^4)x^4+3(1-13\beta^2-4\beta^4)x^2+3\beta^2(-3-5\beta^2)\Big]$$

$$\frac{-x^2}{12(x^2+1)^3(x^2+\beta^2)^3}\Big[3(3+\beta^2)(1-2\beta^2)x^6+(4-21\beta^2-21\beta^4+2\beta^6)x^4$$

$$+(3-39\beta^2-6\beta^4+6\beta^6)x^2+3\beta^2(-3-5\beta^2+4\beta^4)\Big]$$

(4.16)

The solution to the homogeneous problem is then

$$w_c = \left[1+(\gamma-1)M^2\right]\frac{M^2}{4}\left\{\frac{-\frac{\pi}{2}i}{(z+i\beta)^3} + \frac{z}{(z^2+\beta^2)^2}\left[\frac{-(z^2-3\beta^2)\ln\frac{1-iz}{1+\beta}}{z^2+\beta^2}\right.\right.$$

$$\left.\left. + \frac{z^2+\beta(4+5\beta)}{2(1+\beta)^2}\right]\right\} - \frac{(\gamma-1)}{12}\left[\frac{3}{z+i} - \frac{3(z+2i)}{(z+i)^2} + \frac{3z^2+9iz-8}{(z+i)^3} - \frac{3}{\beta}\,\frac{1}{z+i\beta}\right.$$

$$\left. + \frac{3(5-3\beta^2)}{2\beta}\,\frac{z+2i\beta}{(z+i\beta)^2} - \frac{(3-\beta^2)}{2\beta}\,\frac{(3z^2+9iz\beta-8\beta^2)}{(z+i\beta)^3}\right] - \frac{1}{12}\left[\frac{3}{1-\beta^2}\,\frac{1}{z+i}\right.$$

$$- \frac{3(z+2i)}{(z+i)^2} + \frac{3z^2+9iz-8}{(z+i)^3} - \frac{3}{\beta(1-\beta)^2}\,\frac{1}{z+i\beta} + \frac{3(5+2\beta^2)}{2\beta}\,\frac{z+2i\beta}{(z+i\beta)^2}$$

$$\left. - \frac{(3+2\beta^2)}{2\beta}\,\frac{(3z^2+9iz\beta-8\beta^2)}{(z+i\beta)^3}\right] \tag{4.17}$$

and the pressure on the wall may be shown to be:

$$p(x,0) = \left[1+(\gamma-1)M^2\right]\frac{M^2}{4}\left\{\frac{-(3x^2-\beta^2)\tan^{-1}x}{(x^2+\beta^2)^3} + \frac{x}{(x^2+\beta^2)^2}\left[\frac{-\frac{1}{2}(x^2-3\beta^2)\ln\frac{1+x^2}{(1+\beta)^2}}{x^2+\beta^2}\right.\right.$$

$$\left.\left. + \frac{x^2+\beta(4+5\beta)}{2(1+\beta)^2}\right]\right\} + \frac{(\gamma-1)x}{12\beta}(1-\beta)\left[\frac{-3}{x^2+1} + \frac{2}{(x^2+1)^2} - \frac{8}{(x^2+1)^3}\right] + \frac{x}{12\beta}$$

$$\left[\frac{-3}{(1+\beta)(x^2+1)} + \frac{2}{(x^2+1)^2} - \frac{8}{(x^2+1)^3}\right] \tag{4.18}$$

Proceeding as before,

$$u(x,0) = -p(x,0) - \frac{1}{4}\left(\frac{\pi}{2}+\tan^{-1}x\right) - \frac{1}{4}\,\frac{x}{x^2+1} + \frac{1}{6}\,\frac{x}{(x^2+1)^2} - \frac{2}{3}\,\frac{x}{(x^2+1)^3} \tag{4.19}$$

$$s(x,0) = \gamma M^2\left[\frac{1}{4}\left(\frac{\pi}{2}+\tan^{-1}x\right) + \frac{1}{4}\,\frac{x}{(x^2+1)} - \frac{1}{6}\,\frac{x}{(x^2+1)^2} + \frac{2}{3}\,\frac{x}{(x^2+1)^3}\right] \tag{4.20}$$

$$\rho(x,0) = M^2 p - \frac{(\gamma-1)}{\gamma} s$$

(4.21)

These quantities are shown in Figs. 12, 13, 14, and 15 for various values of M and for $\gamma = 1.0$, and $\gamma = 1.4$. Comment on these graphs will also be reserved until comparison can be made with the supersonic case.

It is clear that the algebraic complexity of the subsonic solutions will preclude the use of more complicated magnetic fields. On the other hand, the two problems which have been solved serve to show the trend of the effects of compressibility at subsonic speeds on the type of flow being considered. It is interesting to observe that, as in ordinary linearized compressible flow, the pressure increases with $(1 - M^2)^{-1/2}$.

5. Supersonic Problems

When the flow is supersonic, the problems under consideration can be treated by the method of characteristics. Introducing

$$\xi = \frac{M}{2\beta} x - \frac{M}{2}(y+1), \quad \eta = \frac{M}{2\beta} x + \frac{M}{2}(y+1)$$

(5.1)

$$x = \frac{\beta}{M}(\eta + \xi), \quad y+1 = \frac{1}{M}(\eta - \xi)$$

(5.2)

where, for supersonic flow, $\beta^2 = M^2 - 1$, Eqs. (2.25) and (2.32) become:

$$\frac{\partial}{\partial \xi}[\beta p - v] = -\frac{\beta}{M} b_x b_y + [1 + (\gamma-1)M^2] \frac{1}{M} b_y^2$$

(5.3)

$$\frac{\partial}{\partial \eta}[\beta p + v] = \frac{\beta}{M} b_x b_y + [1 + (\gamma-1)M^2] \frac{1}{M} b_y^2$$

(5.4)

When there is just a single plate at y = 0, the problem can be solved for arbitrary field; the appropriate characteristics are shown in Fig. 16. In order to find the conditions on the plate it is only necessary to integrate Eq. (5. 3) along the characteristic marked 1; since v (x, 0) = 0,

$$P(x,0) = -\frac{1}{M}\int_{-\infty}^{\eta-M} b_x\left\{\frac{\beta}{M}(\eta+t), \frac{\eta-t-M}{M}\right\} b_y\left\{\frac{\beta}{M}(\eta+t), \frac{\eta-t-M}{M}\right\} dt$$

$$+\frac{[1+(\gamma-1)M^2}{M\beta}\int_{-\infty}^{\eta-M} b_y^2\left\{\frac{\beta}{M}(\eta+t), \frac{\eta-t-M}{M}\right\} dt$$

(5. 5)

In order to find conditions anywhere else in the flow field, two additional integrations are required, one from infinity along the characteristic marked 2, and the other from the plate (where v = 0 and p is given by Eq. (5. 5)) along the characteristic marked 3. By way of illustration, the two cases considered in the last section will be worked again for the supersonic case.

Single Wire Case

For the single wire, with the components of the magnetic field given by Eq. (3. 10), the pressure on the wall becomes:

$$P(x,0) = \frac{\frac{1}{2}[1+(\gamma-1)M^2]}{\beta}\left\{\frac{\frac{\pi}{2}\operatorname{sgn}(x+\beta)+\tan^{-1}\frac{\beta x-1}{x+\beta}}{x+\beta}\right\}$$

$$-\frac{\frac{1}{2}[(\gamma-1)\beta x+\gamma]}{\beta(x^2+1)} \qquad (5.6)$$

where sgn $(x + \beta)$ = +1, 0 or -1 according as $(x + \beta)$ is positive, zero or negative. The rise in pressure at transonic Mach numbers is again evident. Proceeding as before,

$$u(x,0) = \frac{-\frac{1}{2}[1+(\gamma-1)M^2]}{\beta}\left\{\frac{\frac{\pi}{2}\operatorname{sgn}(x+\beta)+\tan^{-1}\frac{\beta x-1}{x+\beta}}{x+\beta}\right\} - \frac{1}{2}\left(\frac{\pi}{2}+\tan^{-1}x\right) + \frac{\frac{1}{2}\gamma(\beta x+1)}{\beta(x^2+1)}$$

(5. 7)

$$S(x,0) = \frac{1}{2}\gamma M^2\left[\left(\frac{\pi}{2}+\tan^{-1}x\right) - \frac{x}{x^2+1}\right]$$

(5. 8)

$$\rho(x,0)=\frac{1}{2}\left[1+(\gamma-1)M^2\right]\frac{M^2}{\beta}\left\{\frac{\frac{\pi}{2}\,\mathrm{sgn}(x+\beta)+\tan^{-1}\frac{\beta x-1}{x+\beta}}{x+\beta}\right\}-\frac{\frac{1}{2}\gamma M^2}{\beta(x^2+1)}$$

$$-\frac{1}{2}(\gamma-1)M^2\left(\frac{\pi}{2}+\tan^{-1}x\right)$$

(5.9)

These quantities are shown on Figs. 8, 9, 10 and 11 along with the subsonic solutions. The following points should be noticed in connection with these graphs. First, the curves for $\gamma = 1.0$ may be thought of as emphasizing the force aspects of the flow, while the curves for $\gamma = 1.4$ bring out the effect of heating. This is particularly clear in Figs. 11a and 11b where it will be noticed that the change in density is much greater for the case $M = 2$, $\gamma = 1.4$ than $M = 5$, $\gamma = 1.0$. This may be thought of as being due to the fact that large Mach numbers correspond to low temperatures; in this case conversion of a small fraction of the flow energy into heat is sufficient to cause a large rise in the temperature of the flow accompanied by a large expansion. From 9a and 9b it can be seen that the change in velocity is largely independent of the Mach number, thus the flow energy converted into heat is roughly constant but has a much greater effect on the flow at high Mach numbers. Finally, this phenomena will set a limit on the validity of these calculations as follows: a small change in the temperature will produce a relatively large change in conductivity (σ normally varies as quite a high power of T). Thus, for high Mach numbers $\Delta\ T/T$ cannot exceed some quite small value in order that the assumption of constant conductivity should remain valid.

As far as the pressure effects are concerned, the dominant effect is transonic; a large rise in pressure appears as $M \rightarrow 1$ for both the supersonic and transonic cases; this is a characteristic phenomenon in all compressible flows, and its appearance in this magnetohydrodynamic case is quite interesting.

Linear Dipole

For the linear dipole, the quantities are given by:

$$P(x,0)=\frac{1}{6M^2(x^2+1)^3}(x^3-3\beta x^2-3x+\beta)$$

$$+\frac{\left[1+(\gamma-1)M^2\right]M^2}{4\beta(x+\beta)^3}\left[\tan^{-1}\frac{\beta x-1}{x+\beta}+\frac{\pi}{2}\,\mathrm{sgn}(x+\beta)+\frac{(x+\beta)(\beta x-1)}{M^2(x^2+1)}\right]$$

$$+\frac{\left[1+(\gamma-1)M^2\right]}{6M^2\beta(x^2+1)^3}(-\beta x^3-3x^2+3\beta x+1)$$

(5.10)

$$u(x,0) = -p(x,0) - \frac{1}{4}\left(\frac{\pi}{2} + \tan^{-1}x\right) - \frac{1}{4}\frac{x}{x^2+1} + \frac{1}{6}\frac{x}{(x^2+1)^2} - \frac{2}{3}\frac{x}{(x^2+1)^3}$$

(5.11)

$$s = \gamma M^2 \; \frac{1}{4}\left(\frac{\pi}{2} + \tan^{-1}x\right) + \frac{1}{4}\frac{x}{x^2+1} - \frac{1}{6}\frac{x}{(x^2+1)^2} + \frac{2}{3}\frac{x}{(x^2+1)^3}$$

(5.12)

$$\rho = M^2 p - \frac{(\gamma-1)}{\gamma} s$$

(5.13)

and these are again shown on Figs. 12, 13, 14 and 15. The conclusions to be drawn from these graphs are broadly similar to those for the case of the single wire. Once again the changes in velocity are largely independent of Mach numbers, but high Mach numbers again result in large expansion for $\gamma = 1.4$. Note that in Fig. 15a the density change for M = 5 is given while in Fig. 15b M = 2 is the highest value plotted. As far as the pressure is concerned, the dominant effect is again transonic; a large rise in pressure appears as $M \to 1$.

Channel flow problems in the supersonic case are more difficult in view of the multiple reflections, but a solution in the form of an infinite series might possibly be constructed. No such cases have been attempted.

6. Variable Conductivity Problems

Apart from problems involving physical boundaries, an interesting and in many ways realistic problem is that in which the conductivity is allowed to vary with position. Up to this point, it has been assumed that the conductivity σ^* was a constant at all points. It is possible by using the methods that have been developed in this paper to treat problems in which σ^* is assumed to be a function of y only. Attention will be restricted to the incompressible case with a single wall at y = 0; the extension to channel flows, or to subsonic or supersonic flows is immediate.

It is necessary first to modify the equations used to describe the flow to include the possibility of variable conductivity. Taking σ^* as a reference value of the conductivity, let σ' be the actual value as a function of position, and define $\sigma = \sigma'/\sigma^*$. Then, Eqs. (2.11) to (2.15) will be modified by replacing S with σS. Hence, using Eqs. (2.19) to (2.23), the right-hand sides of Eqs. (2.24), (2.25), and (2.27) should be multiplied by σ. In particular, the equations governing p and v become

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} = \sigma b_x b_y \tag{6.1}$$

$$\frac{\partial v}{\partial y} - (1-M^2)\frac{\partial p}{\partial x} = \sigma[1+(\gamma-1)M^2] b_y^2 \tag{6.2}$$

For the incompressible case it is only necessary to set M = 0.

The simplest problem in which σ is not constant is where $\sigma = 1$ for $0 < y < H$ and $\sigma = 0$ for $y > H$. The appropriate boundary conditions at $y = H$ are that v and p should be continuous. There will, however, be a jump in u. Let subscript 1 refer to the region $0 < y < H$, and subscript 2 refer to the region $y > H$. Then the solution can be constructed as follows; first, in region 1, take the particular solution given by Eqs. (3.4) and (3.5). This may be referred to as (v_{P1}, p_{P1}). Next, define the complex function

$$F(x) = p_{P_1}(x,H) + i v_{P_1}(x,H) \tag{6.3}$$

It is now required to find a pair of complex functions, $w_1(z)$ and $w_2(z)$ such that $w_1(z)$ is analytic in the region $0 < y < H$ while $w_2(z)$ is analytic in the entire region $y > H$, and such that

$$w_2(x+iH) = w_1(x+iH) + F(x) \tag{6.4}$$

However, it is convenient to require that w_1 should also be analytic in the region $y < 0$. In this case the general solution to this problem is

$$w_1(z) = -\frac{1}{2\pi i}\int_{\mathcal{C}_1} \frac{F(\zeta)}{\zeta - z}\, d\zeta \tag{6.5}$$

$$w_2(z) = -\frac{1}{2\pi i}\int_{\mathcal{C}_2} \frac{F(\zeta)}{\zeta - z}\, d\zeta \tag{6.6}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are large semicircles having the line $y = H$ for diameter and enclosing respectively the regions $y < H$ and $y > H$. In simple cases it will be possible to find the functions w_1 and w_2 by inspection, but formulae (6.5) and (6.6) are quite general. At this stage the solution in region 1 may be written symbolically as $(p_{P1}, v_{P1}) + w_1$, while in region 2 the solution is w_2. The conditions of continuity across the line $y = H$ have been met, but the boundary at $y = 0$ has not yet been introduced. In order to take account of this, a function $w(z)$ is introduced. $w(z)$ is analytic in the region $y > 0$ and satisfies:

$$\Im w + v_{P1} + \Im w_1 = 0 \quad \text{on } y = 0 \tag{6.7}$$

The final solution is now given by:

$$(p_{P1}, v_{P1}) + w_1 + w \qquad 0 < y < H$$

$$w_2 + w \qquad y > H$$

Single Wire Case

To illustrate the use of this technique, consider first the case of a single wire, the components of the magnetic field being given by Eq. (3.10). The solution (p_{P1}, v_{P1}) is given by Eqs. (3.11) and (3.12), and the function $F(x)$ by

$$F(x) = \frac{\frac{1}{2}\left[x - (H+1)\tan^{-1}\frac{x}{H+1}\right] - \frac{1}{2}\, i \tan^{-1}\frac{x}{H+1}}{x^2 + (H+1)^2} \tag{6.8}$$

The functions w_1 and w_2 may be found by inspection to be:

$$w_1(z) = \frac{\frac{1}{4}\ln\frac{2H+1+iz}{2(H+1)}}{z+i} - \frac{\frac{1}{4}}{z - i(2H+1)} \tag{6.9}$$

$$w_2(z) = \frac{\frac{1}{4}\ln\frac{1-iz}{2(H+1)}}{z+i} + \frac{\frac{1}{4}}{z+i} \tag{6.10}$$

The condition on w at $y = 0$ is now

$$v = \frac{\frac{1}{2}x\tan^{-1}x}{x^2+1} + \frac{\frac{1}{8}\ln\frac{x^2+(2H+1)^2}{4(H+1)^2} - \frac{1}{4}x\tan^{-1}\frac{x}{2H+1}}{x^2+1} + \frac{1}{4}\,\frac{2H+1}{x^2+(2H+1)^2} \tag{6.11}$$

so that

$$w = \frac{\frac{1}{4}\ln\frac{2H+1-iz}{2(H+1)}}{z-i} - \frac{\frac{1}{2}z\ln\frac{1-iz}{2}}{z^2+1} - \frac{\frac{1}{4}}{z+i(2H+1)} \tag{6.12}$$

completing the solution of the problem. The pressure on the wall is given by

$$P(x,0) = \frac{1}{2}x\left[\frac{1}{x^2+1} - \frac{1}{x^2+(2H+1)^2}\right] + \frac{\frac{1}{4}x\ln\left(\frac{x^2+(2H+1)^2}{(H+1)^2(x^2+1)}\right) - \frac{1}{2}\left[\tan^{-1}x - \tan^{-1}\frac{x}{2H+1}\right]}{x^2+1} \tag{6.13}$$

and is shown in Fig. 17. Clearly, when H vanishes and the flow is non-conducting everywhere, all the flow perturbations vanish. However, as may be seen from Fig. 17, there is only a small difference between the case where the flow is conducting only out to y = 1 and the case where it is conducting everywhere.

The jump in u at y = H is given by

$$u(y=H+) - u(y=H-) = \frac{1}{2(H+1)}\left[\frac{\pi}{2} + \tan^{-1}\frac{x}{H+1}\right] - \frac{\frac{1}{2}x}{x^2+(H+1)^2} \tag{6.14}$$

which is, as might be expected, always positive.

Linear Dipole

For the linear dipole case, with t[illegible] omponents of the magnetic field given by Eq. (3.33),

$$F(x) = \frac{\frac{1}{2}x\left[\frac{1}{3}x^2+(H+1)^2\right] + \frac{1}{2}i(H+1)\left[x^2-(H+1)^2\right]}{\left[x^2+(H+1)^2\right]^3} \tag{6.15}$$

Then:

$$w_1 = \frac{1}{48(H+1)^2}\left[\frac{3z^2-6iz(2H+1)-(16H^2+20H+7)}{[z-i(2H+1)]^3}\right] \tag{6.16}$$

$$w_2 = \frac{1}{48(H+1)^2}\left[\frac{3z^2+6iz(H+2)-(8H^2+22H+17)}{(z+i)^3}\right]$$

(6.17)

so that the condition on w is:

$$v = \frac{-\frac{1}{2}(x^2-1)}{(x^2+1)^3} - \frac{(2H+1)}{48(H+1)^2}\left[\frac{3x^4+6(2H^2-1)x^2+(2H+1)^2(16H^2+20H+7)}{[x^2+(2H+1)^2]^3}\right]$$

(6.18)

giving

$$w = \frac{-(z^2+3iz-4)}{8(z+i)^3} + \frac{3z^2+6(2H+1)iz-(16H+20H+7)}{48(H+1)^2[z+i(2H+1)]^3}$$

(6.19)

The pressure on the wall is thus:

$$p(x,0) = \frac{-x(3x^4+2x^2+15)}{24(x^2+1)^3} + \frac{[3x^4+2(10H^2+8H+1)x^2+3(2H+1)^2(8H+12H+5)]}{24(H+1)^2[x^2+(2H+1)^2]^3}$$

(6.20)

and is shown in Fig. 18. Here again it may be seen that the bulk of the perturbation to the pressure on the wall is induced very close to the wall. The magnetic field for this case falls off with distance faster than in the single wire case, and as a result, if the flow is conducting only out to about $y = 0.2$, the effect on the wall pressure is hardly different from the case in which the flow is conducting everywhere.

To conclude this section it will be shown how to adapt solutions of the type given above to the case where the conductivity σ is an arbitrary function of y. Define first $p_H(x, y, H)$, $v_H(x, y, H)$ to be the solution to a problem with a given magnetic field in which $\sigma = 1$ for $y < H$ and $\sigma = 0$ for $y > H$. These are understood to have different forms according to the sign of $(y - H)$. When H is infinite, p_H and v_H stand for the solution to the problem where the conductivity is everywhere unity. The solution to the problem in which σ is a given function of y is then:

$$\left\{p(x,y), v(x,y)\right\} = \sigma(\infty)\left\{p_H(x,y,\infty), v_H(x,y,\infty)\right\}$$

$$-\int_0^\infty \left\{p_H(x,y,H), v_H(x,y,H)\right\} \frac{d\sigma(H)}{dH}\, dH$$

(6.21)

To prove this, note first that all the solutions $p_H(x, y, H)$, $v_H(x, y, H)$ are continuous throughout the plane and give vanishing normal velocity at the wall. Hence, the integrated solution, Eq. (6.21), has the same properties. It remains to show that the integrated solution satisfies the appropriate differential equation. Taking Eq. (6.1) for example,

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} = \sigma(\infty)\left\{\frac{\partial v_H}{\partial x}(x,y,\infty) + \frac{\partial p_H}{\partial y}(x,y,\infty)\right\}$$

$$-\int_0^\infty \left\{\frac{\partial v_H}{\partial x}(x,y,H) + \frac{\partial p_H}{\partial y}(x,y,H)\right\}\frac{d\sigma(H)}{dH}\,dH \tag{6.22}$$

But

$$\frac{\partial v_H}{\partial x} + \frac{\partial p_H}{\partial y} = \begin{cases} b_x b_y & (H > y) \\ 0 & (H < y) \end{cases} \tag{6.23}$$

Thus

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} = \sigma(\infty)\, b_x b_y - \int_y^\infty b_x b_y \frac{d\sigma(H)}{dH}\,dH = \sigma(y)\, b_x b_y \tag{6.24}$$

as required; similarly for Eq. (6.2).

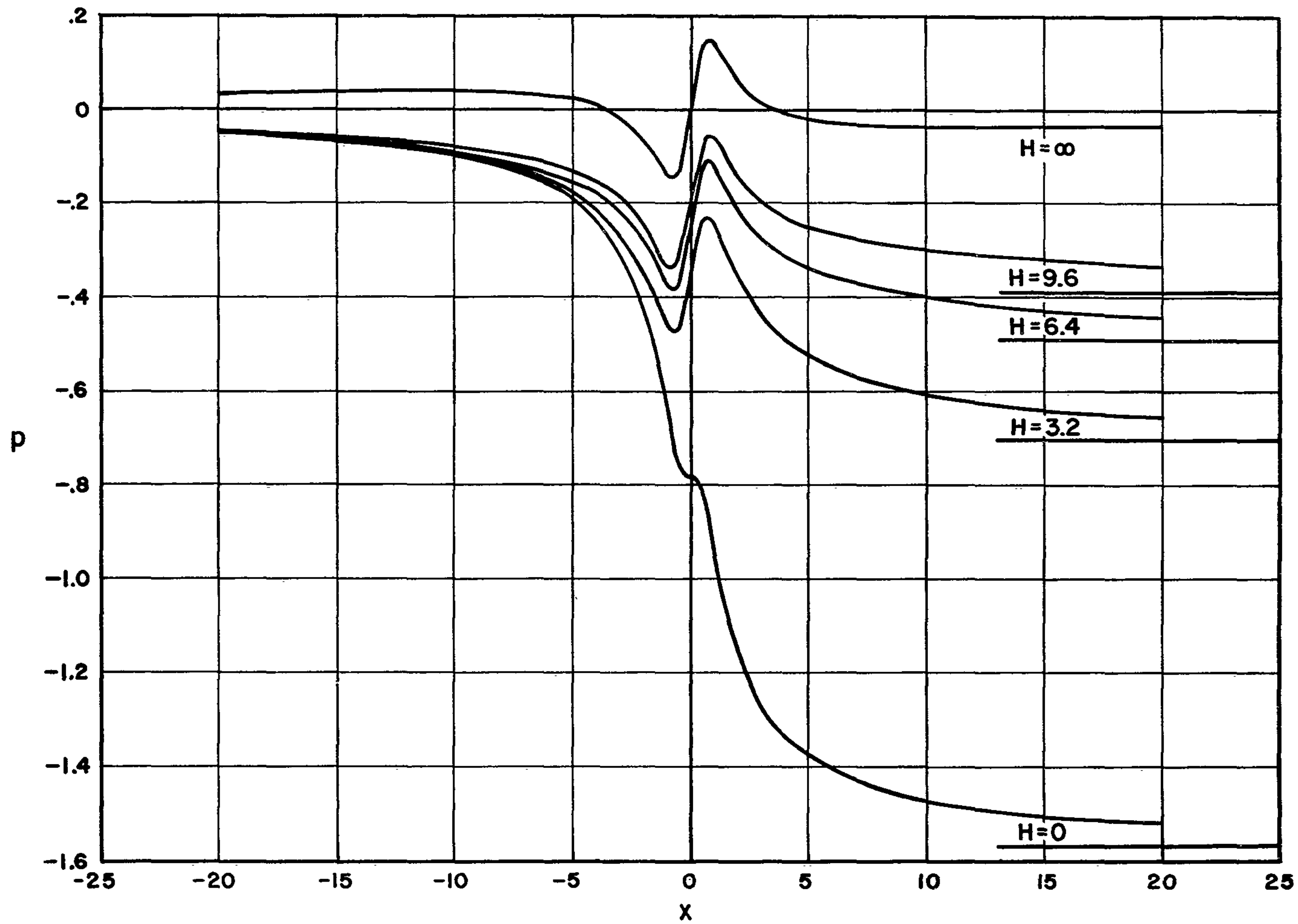

Fig. 1 This is the pressure on the lower wall of a channel of height H when the flow is impeded by the magnetic field due to a current flowing in a single wire at unit distance below the wall.

7194

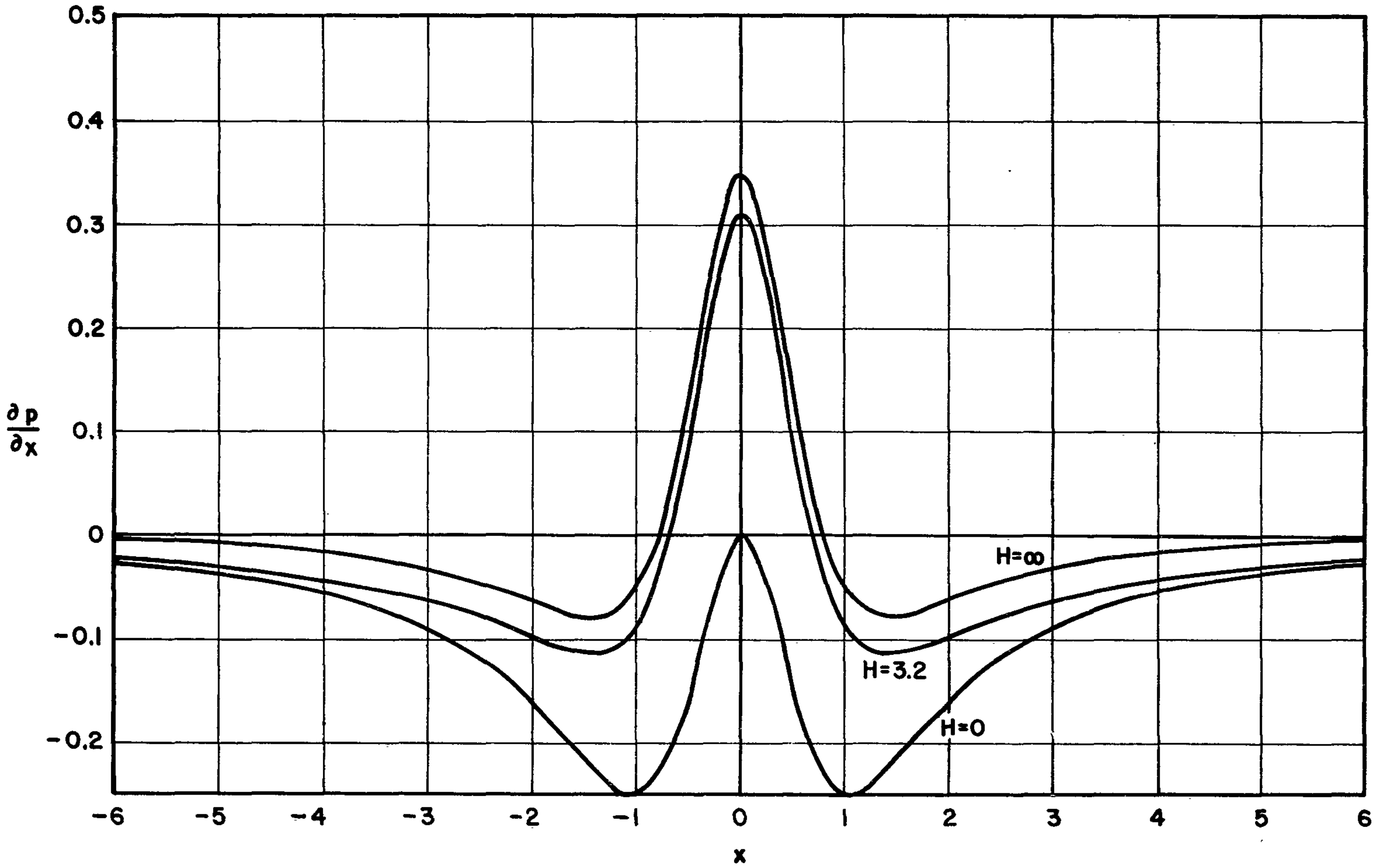

Fig. 2 This is the pressure gradient on the lower wall of a channel of height H when the flow is impeded by the magnetic field due to a current flowing in a single wire at unit distance below the wall.

7195

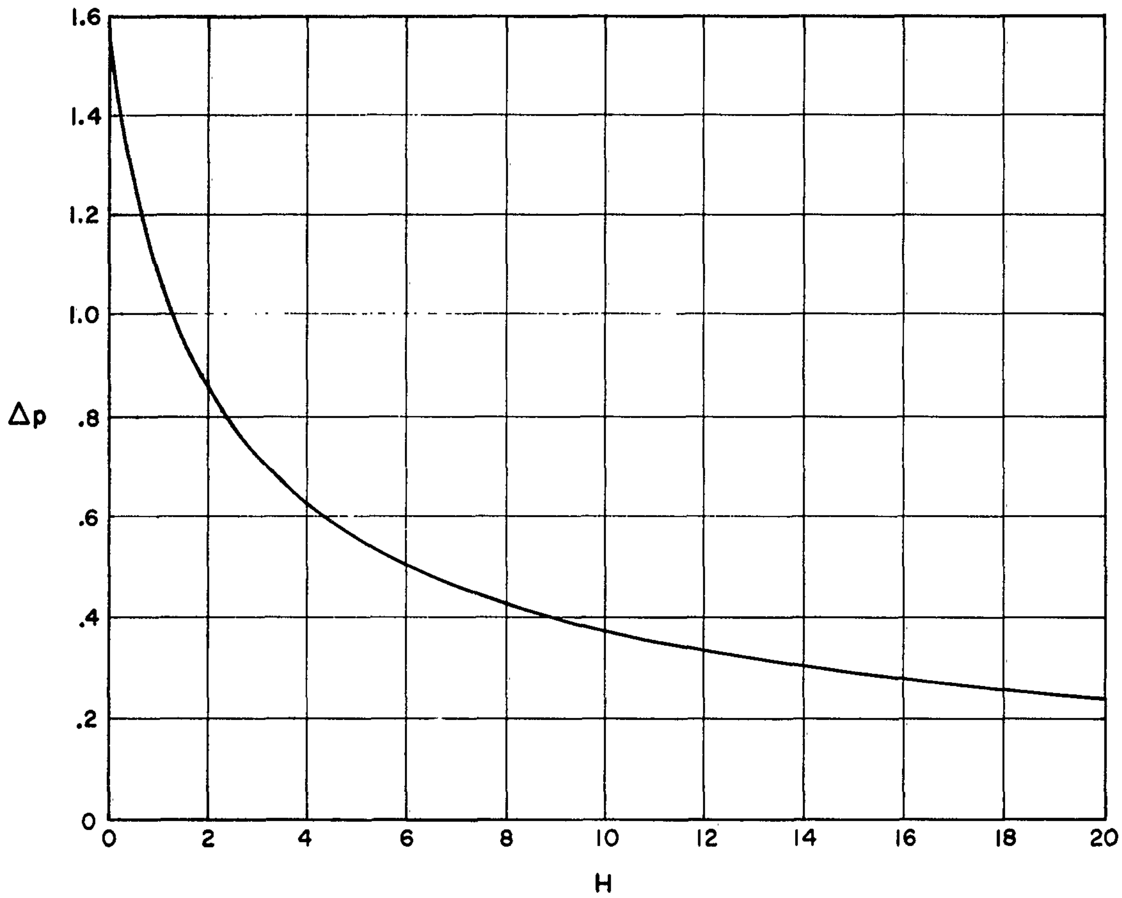

Fig. 3 This is the total pressure drop along a channel of height H when the flow is impeded by the magnetic field due to a current flowing in a single wire at unit distance below the wall.

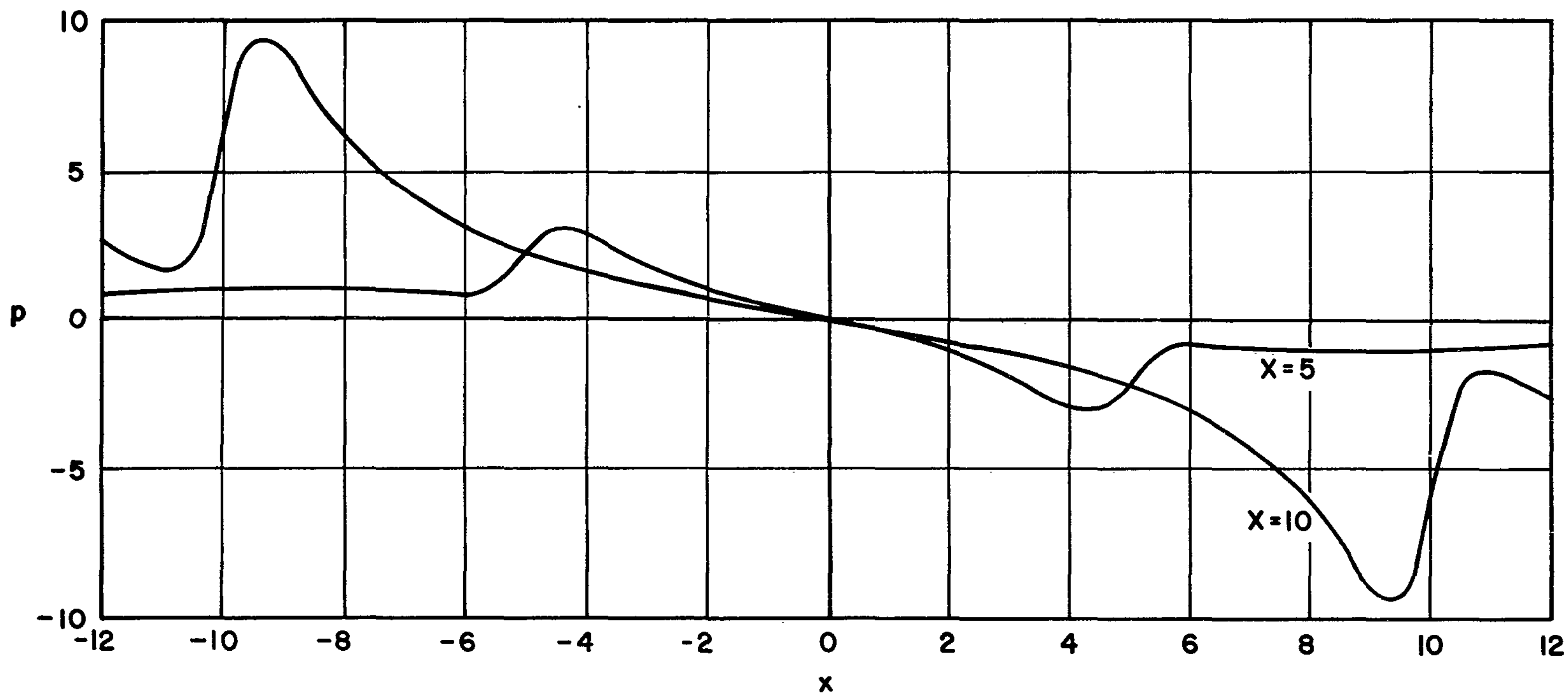

Fig. 4a This is the pressure on the wall when the flow is impeded by the magnetic field due to two wires carrying equal and opposite currents situated unit distance below the wall and 2X from one another. The strength of the current in either wire is varied with X in such a way that the field strength at the origin remains fixed.

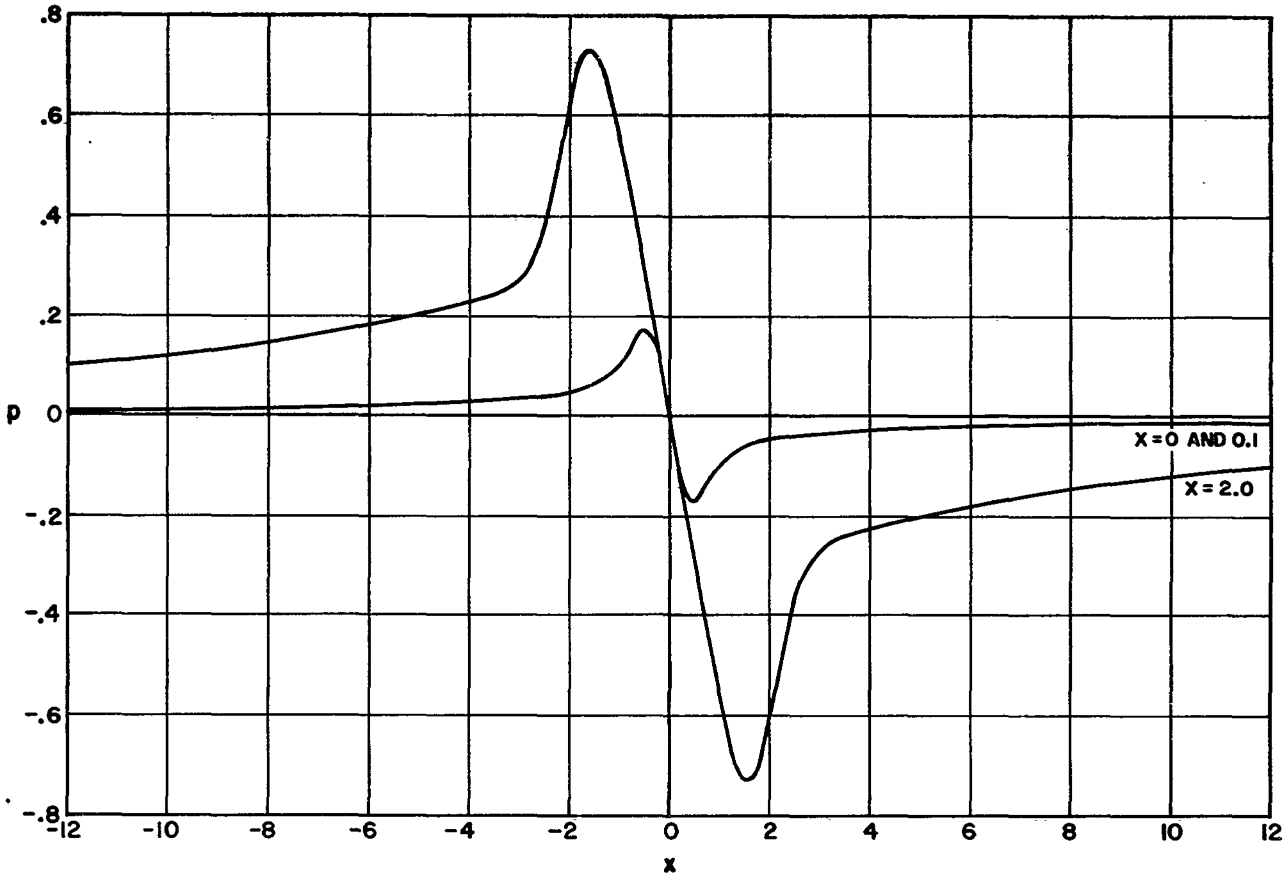

Fig. 4b This is the pressure on the wall when the flow is impeded by the magnetic field due to two wires carrying equal and opposite currents situated unit distance below the wall and 2X from one another. The strength of the current in either wire is varied with X in such a way that the field strength at the origin remains fixed.

7198

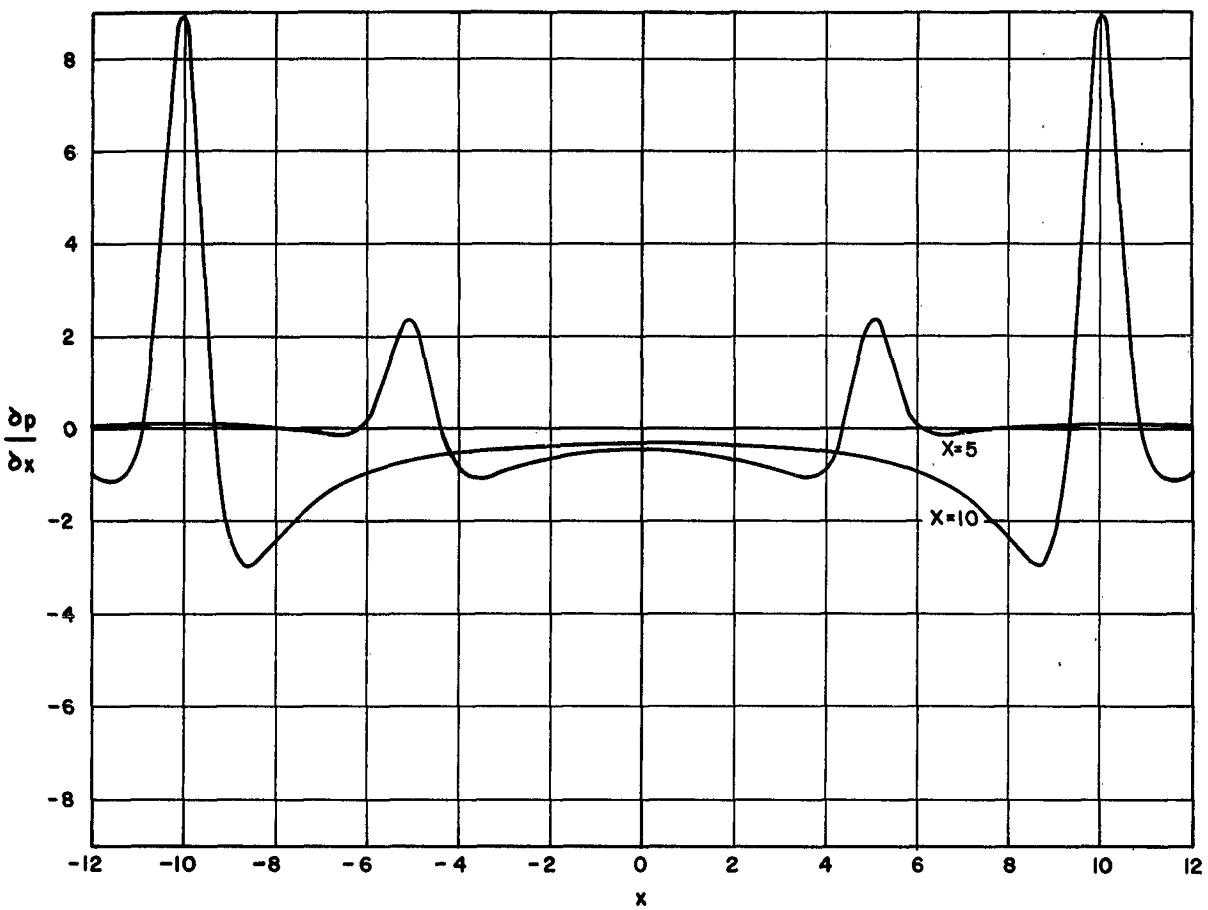

Fig. 5a This is the pressure gradient on the wall when the flow is impeded by the magnetic field due to two wires carrying equal and opposite currents situated unit distance below the wall and 2X from one another. The strength of the current in either wire is varied with X in such a way that the field strength at the origin remains fixed.

7199

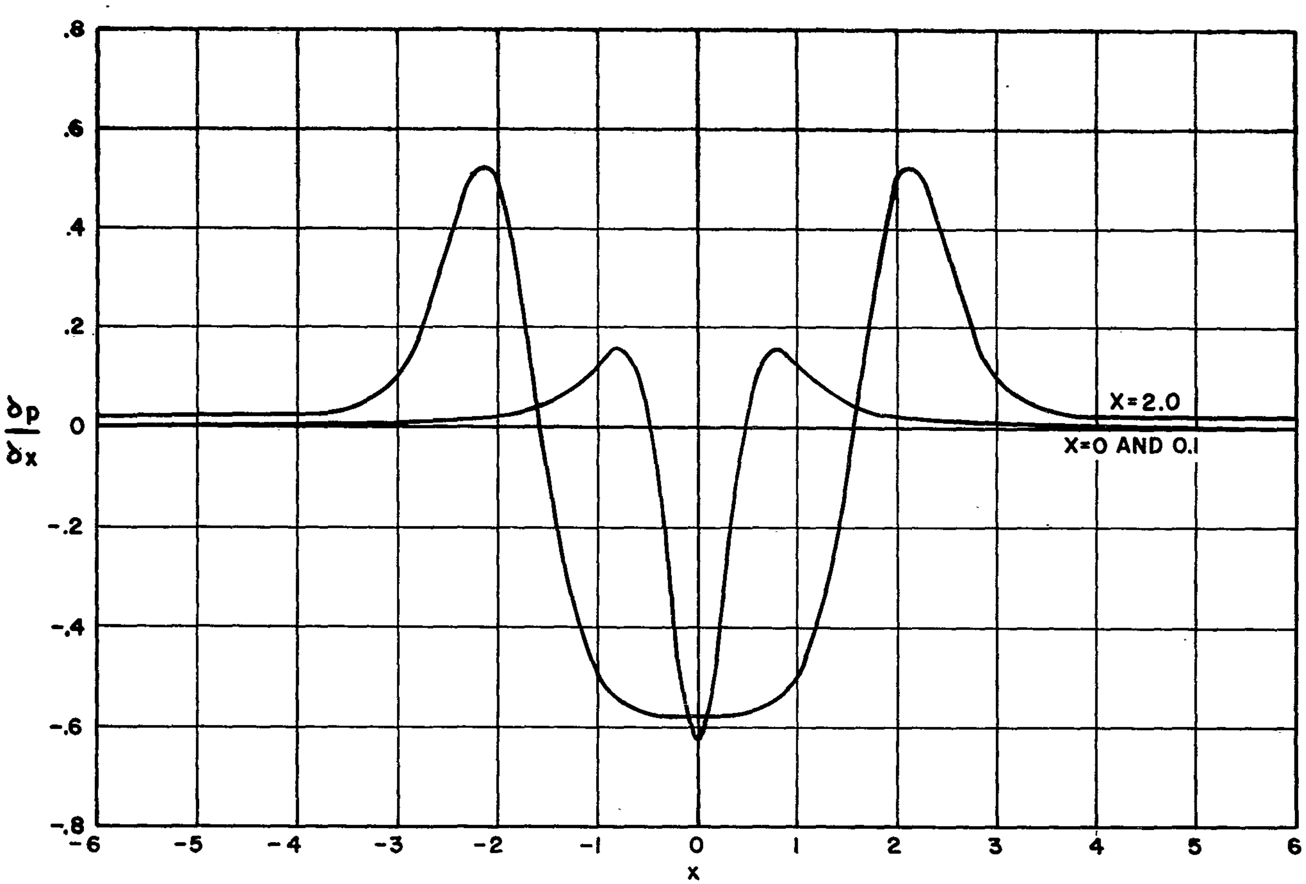

Fig. 5b This is the pressure gradient on the wall when the flow is impeded by the magnetic field due to two wires carrying equal and opposite currents situated unit distance below the wall and 2X from one another. The strength of the current in either wire is varied with X in such a way that the field strength at the origin remains fixed.

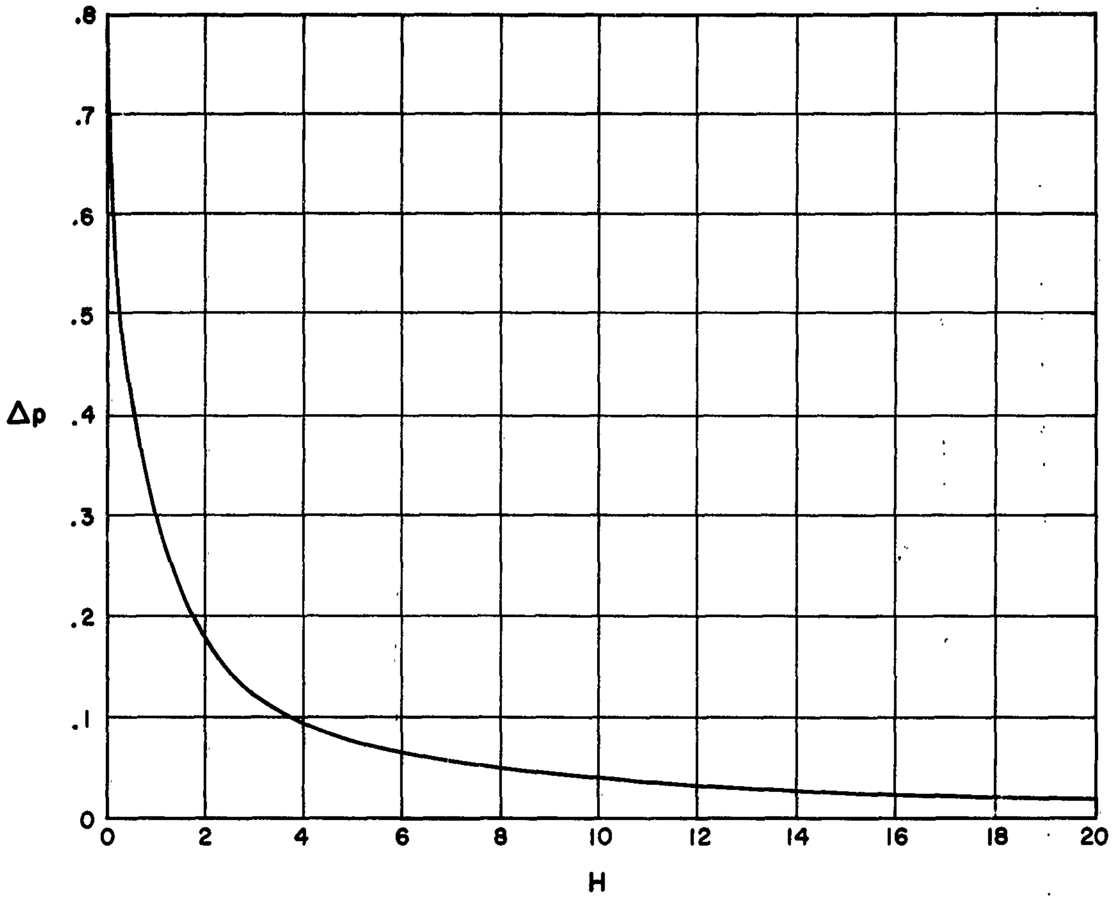

Fig. 6 This is the total pressure drop along a channel of height H when the flow is impeded by the magnetic field due to a linear dipole at unit distance below the wall.

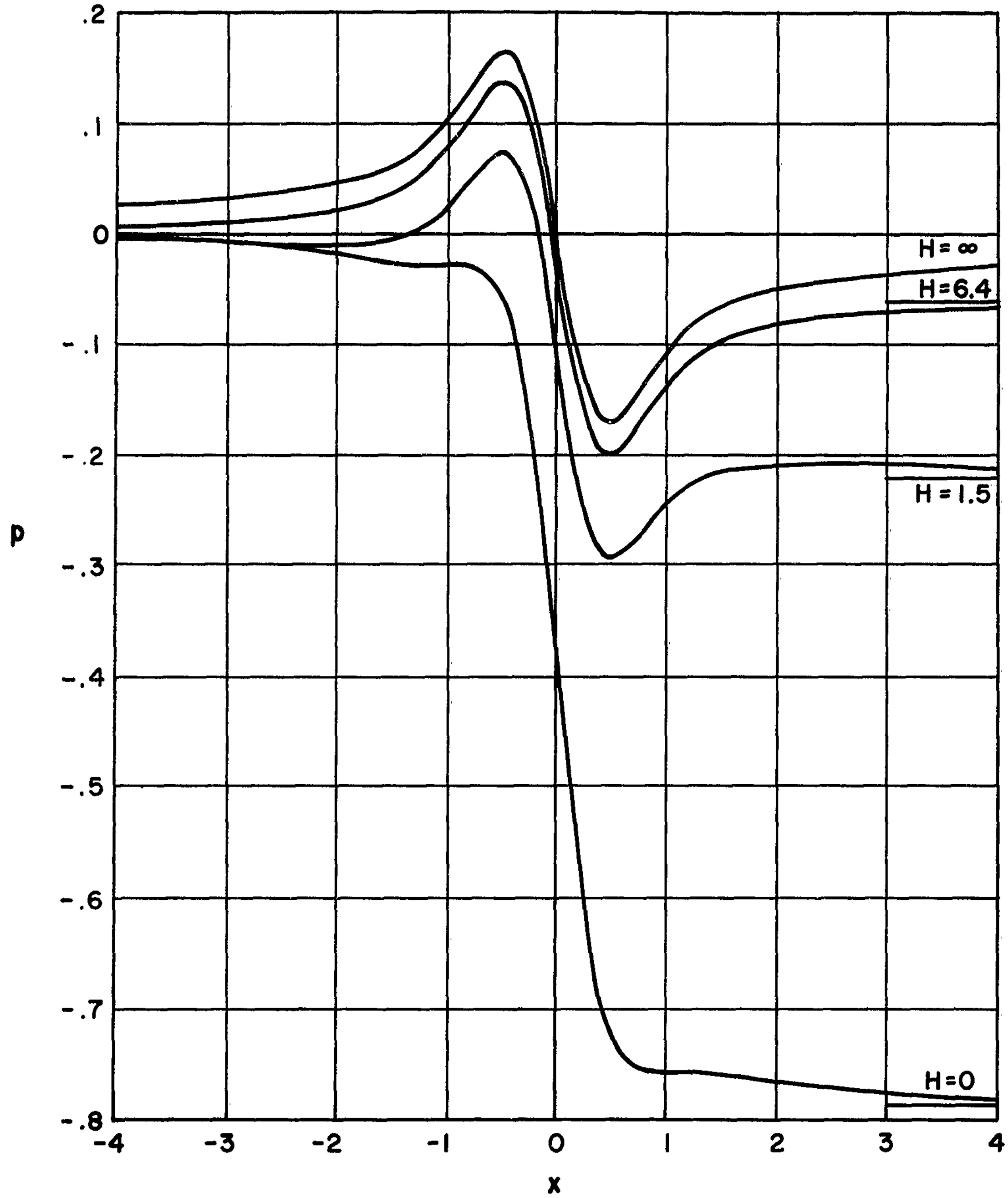

Fig. 7 This is the pressure on the lower wall of a channel of a height H when the flow is impeded by the magnetic field due to a linear dipole at unit distance below the wall.

7202

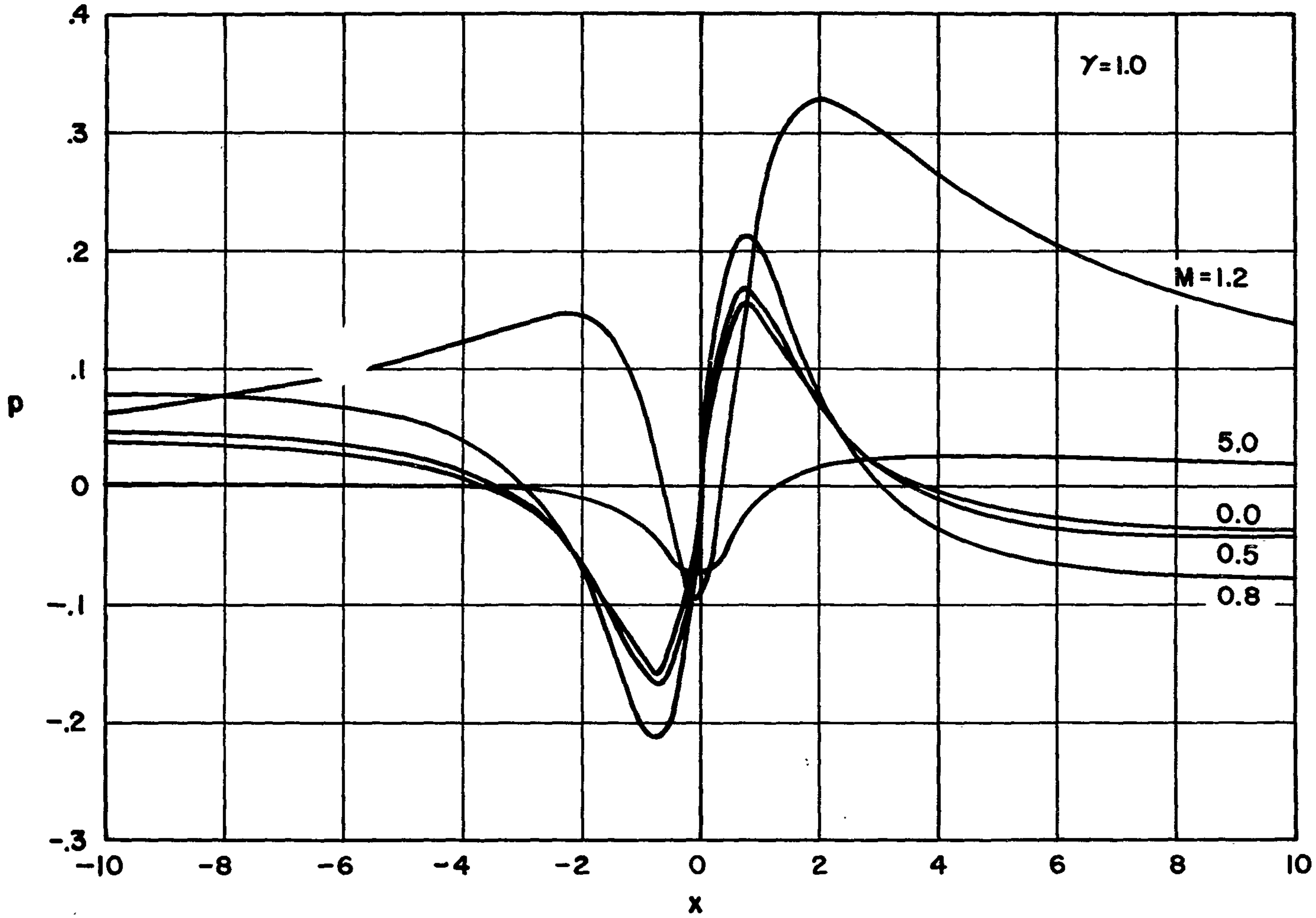

Fig. 8a This is the pressure on a wall when the flow is impeded by the magnetic field due to a current flowing in a single wire at unit distance below the wall. $\gamma = 1.0$.

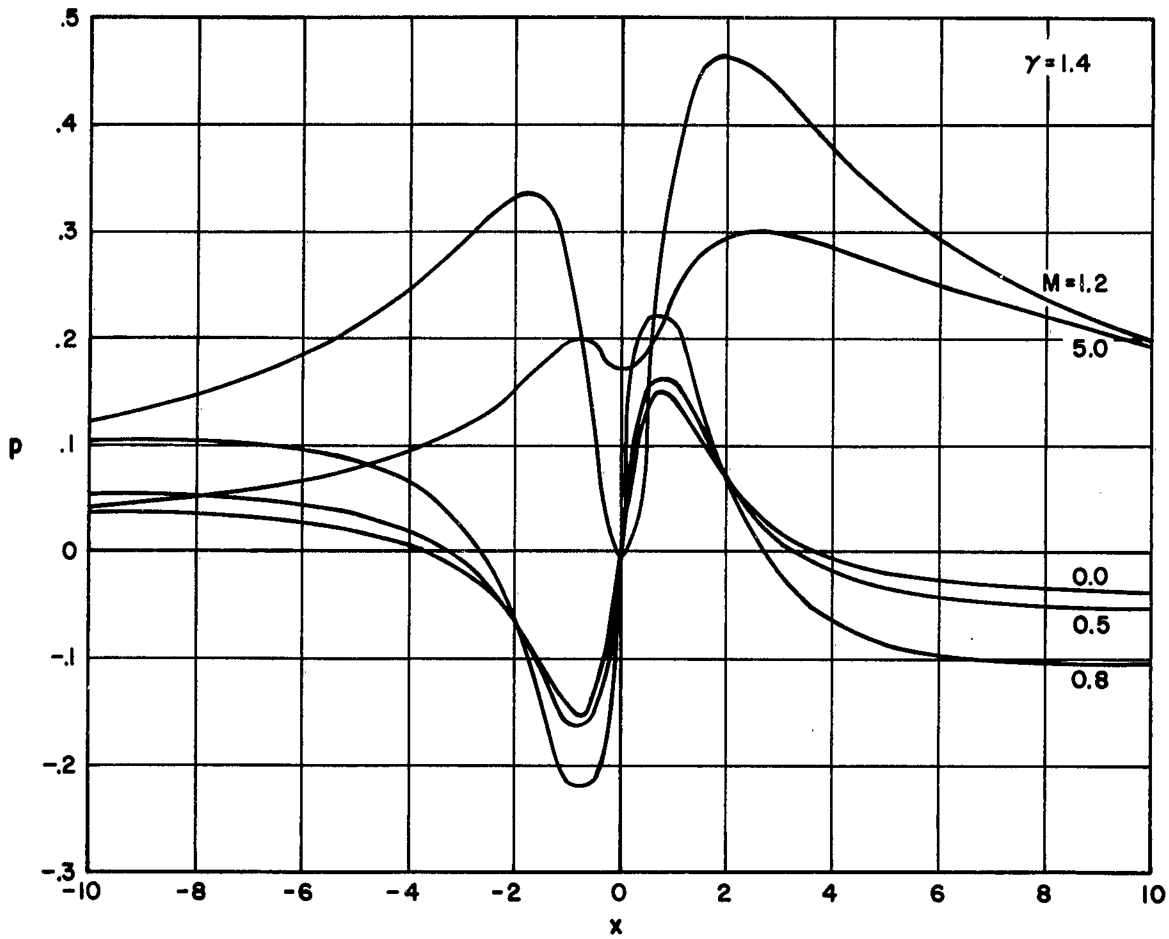

Fig. 8b This is the pressure on a wall when the flow is impeded by the magnetic field due to a current flowing in a single wire at unit distance below the wall. $\gamma = 1.4$.

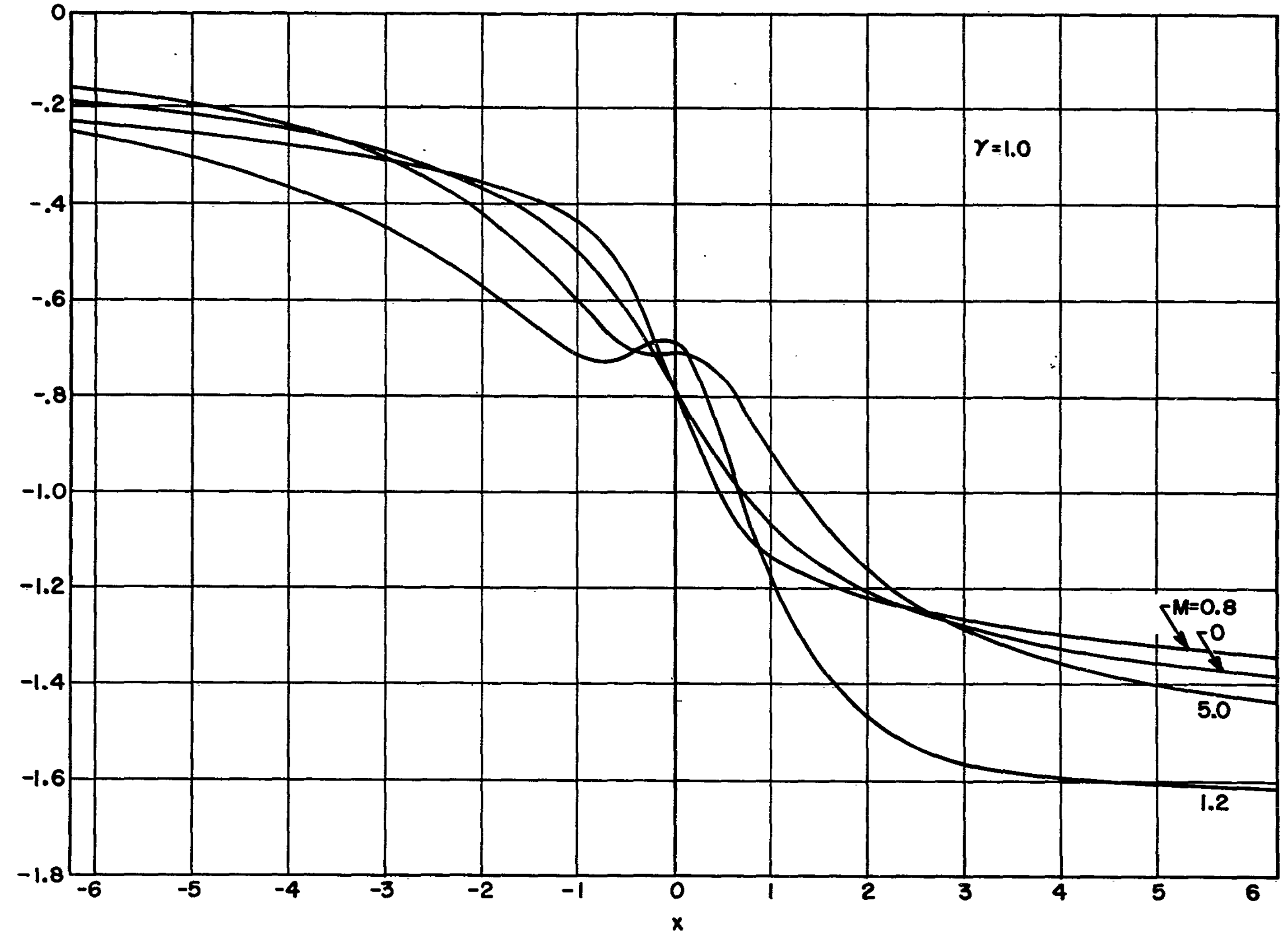

Fig. 9a This is the tangential velocity at the wall when the flow is impeded by the magnetic field due to a current flowing in a single wire at unit distance below the wall. $\gamma = 1.0$.

7205

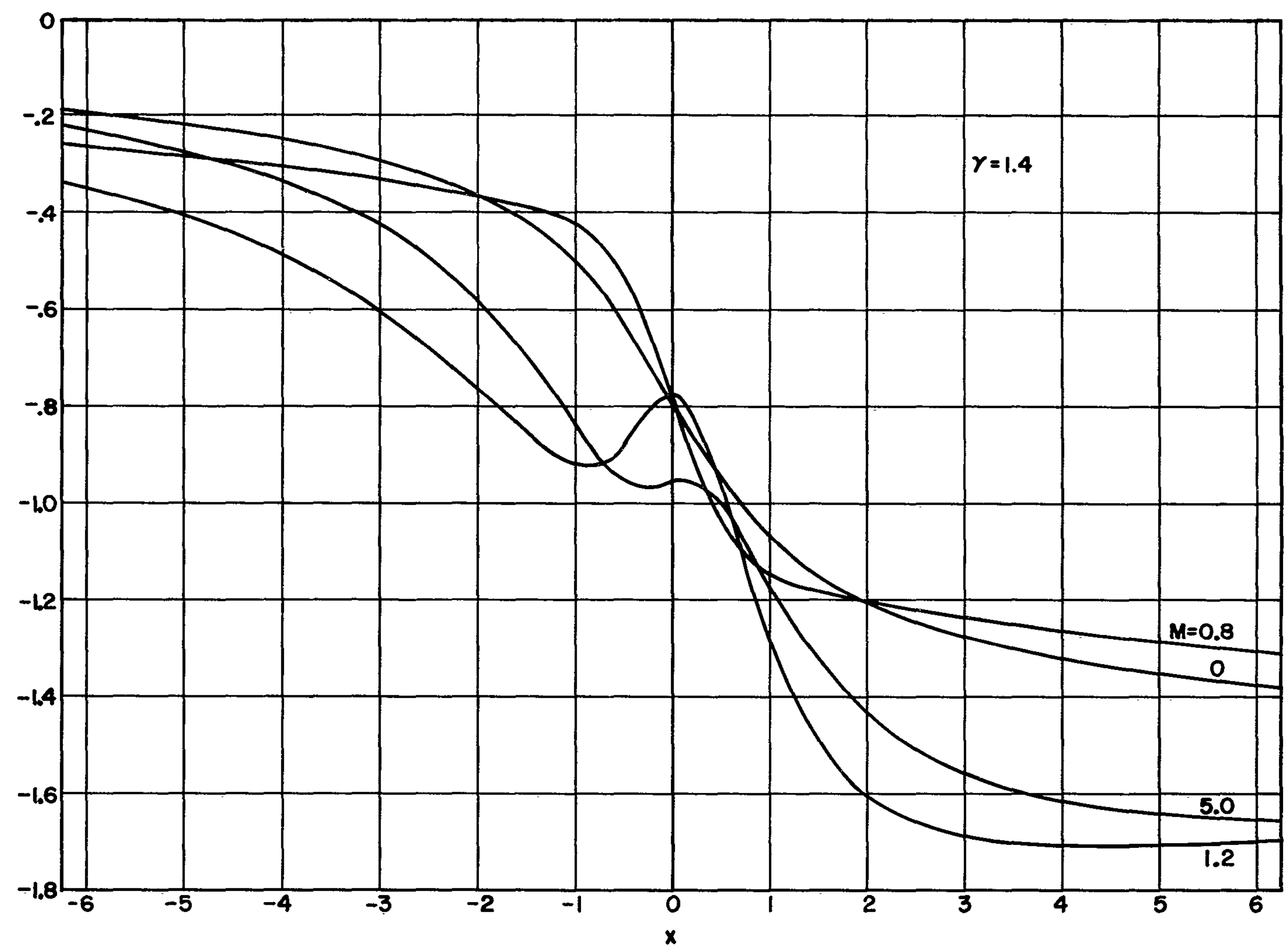

Fig. 9b This is the tangential velocity at the wall when the flow is impeded by the magnetic field due to a current flowing in a single wire at unit distance below the wall. $\gamma = 1.4$.

7206

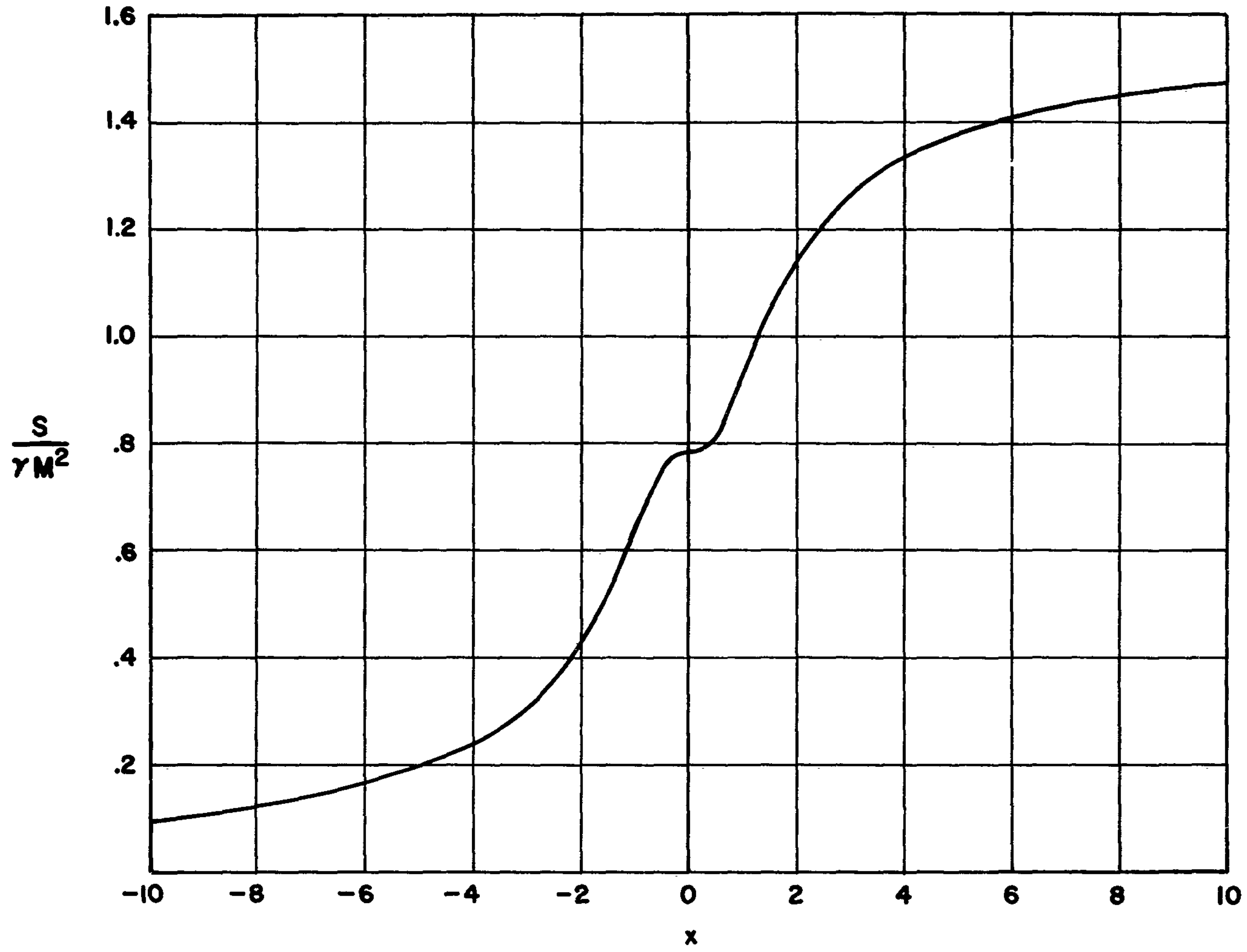

Fig. 10   This is the entropy along a wall when the flow is impeded by the magnetic field due to a current flowing in a single wire at unit distance below the wall.

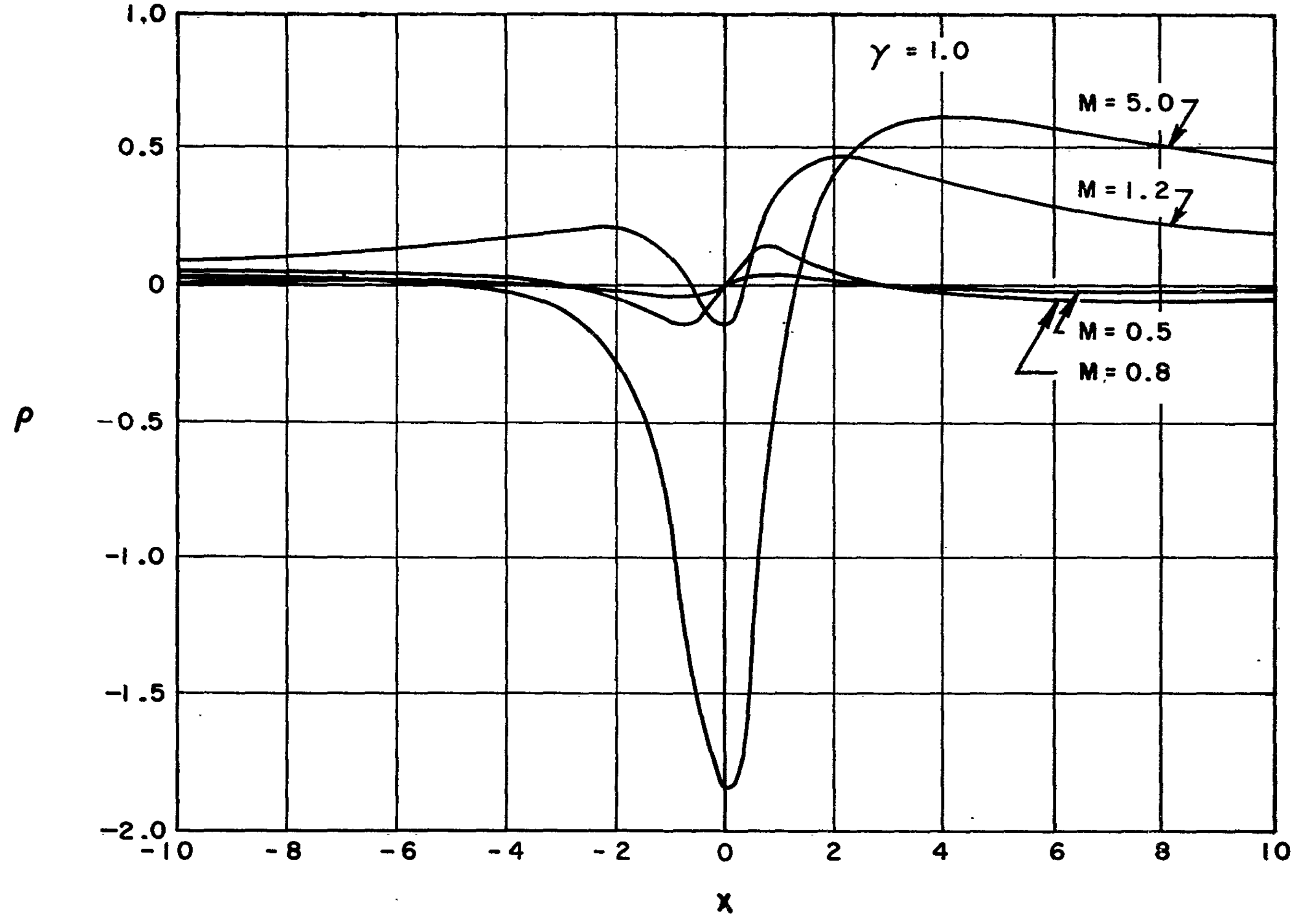

Fig. 11a This is the density on a wall when the flow is impeded by the magnetic field due to a current flowing in a single wire at unit distance below the wall. $\gamma = 1.0$.

7208

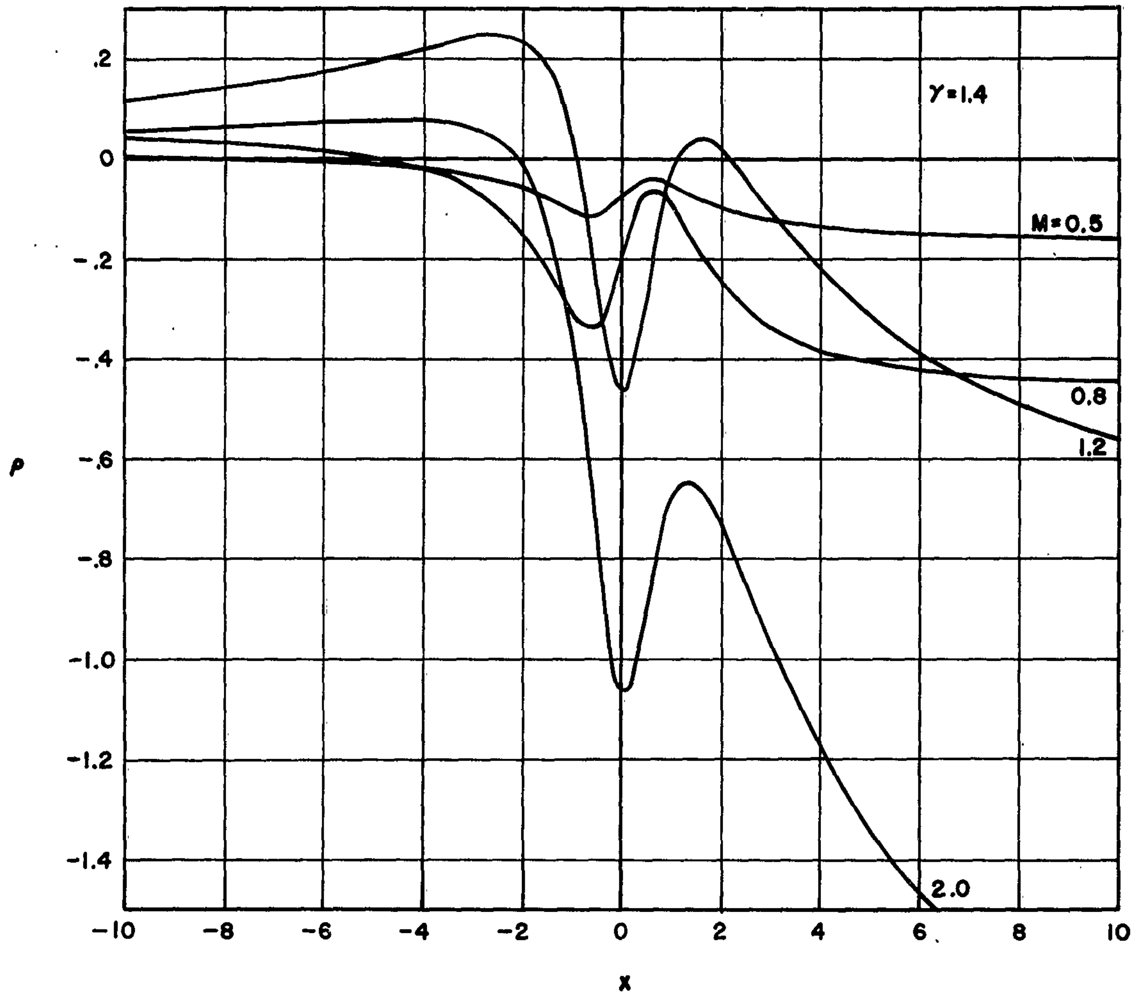

Fig. 11b This is the density on a wall when the flow is impeded by the magnetic field due to a current flowing in a single wire at unit distance below the wall. $\gamma = 1.4$.

7209

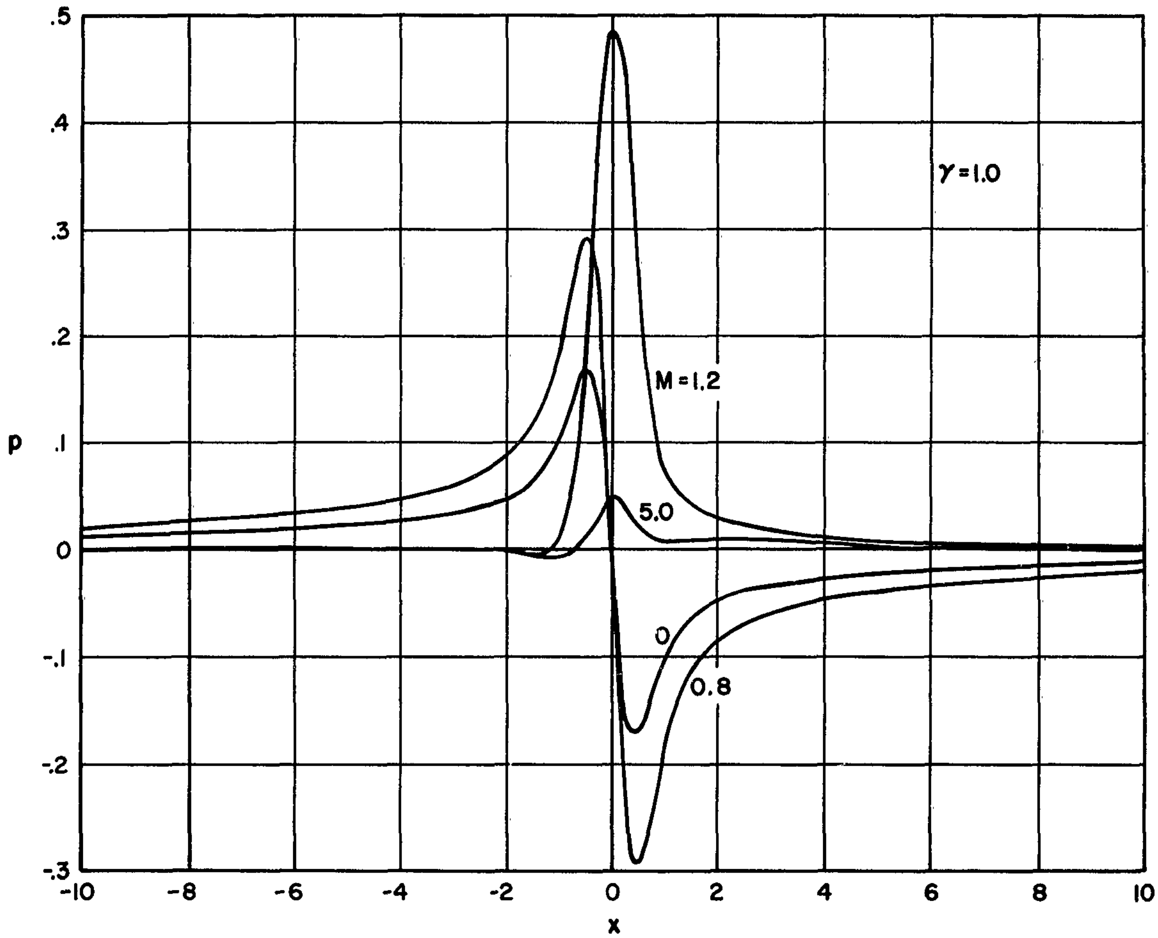

Fig. 12a This is the pressure on the wall when the flow is impeded by the magnetic field due to a linear dipole at unit distance below the wall. $\gamma = 1.0$.

7210

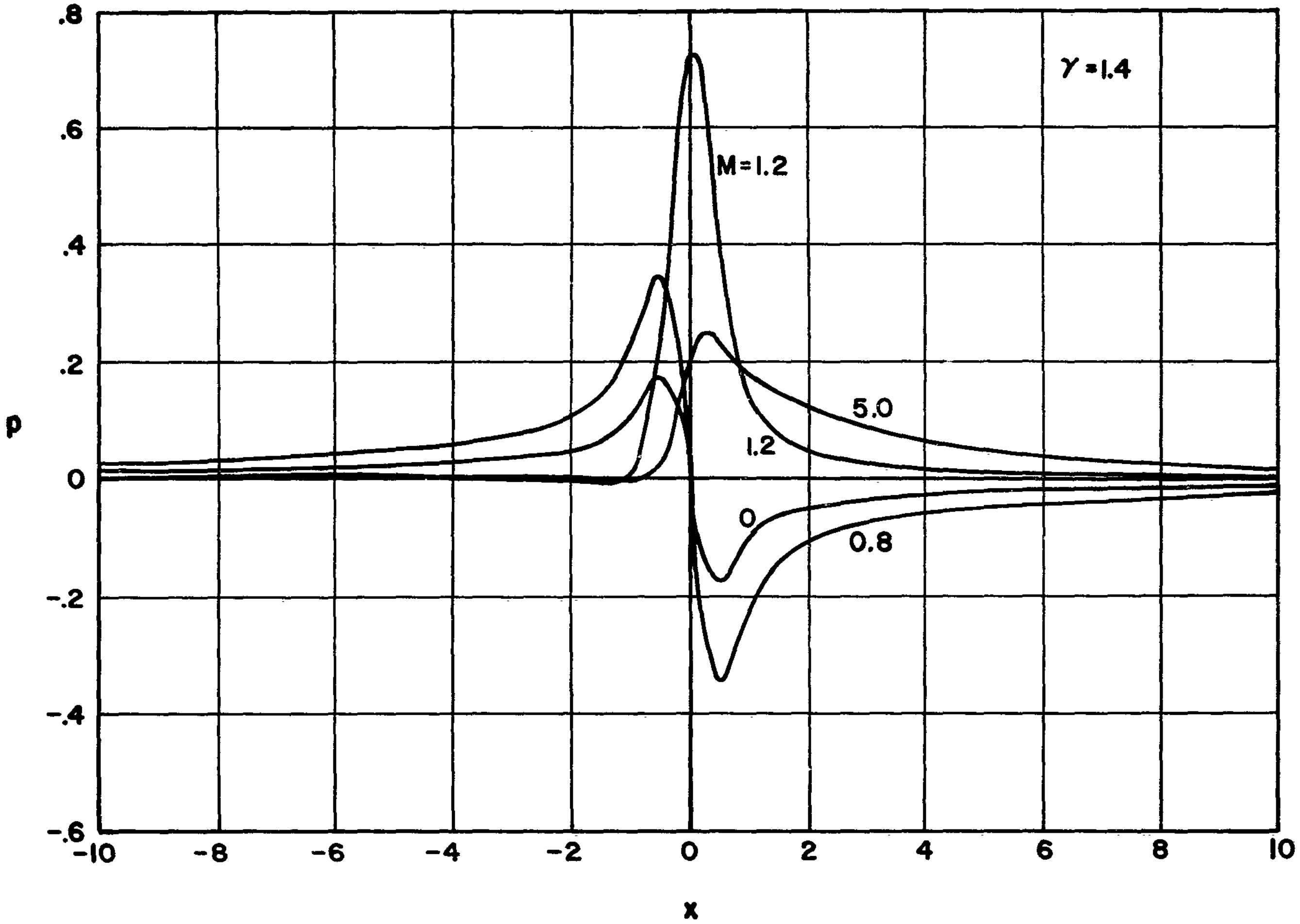

Fig. 12b This is the pressure on the wall when the flow is impeded by the magnetic field due to a linear dipole at unit distance below the wall. $\gamma = 1.4$.

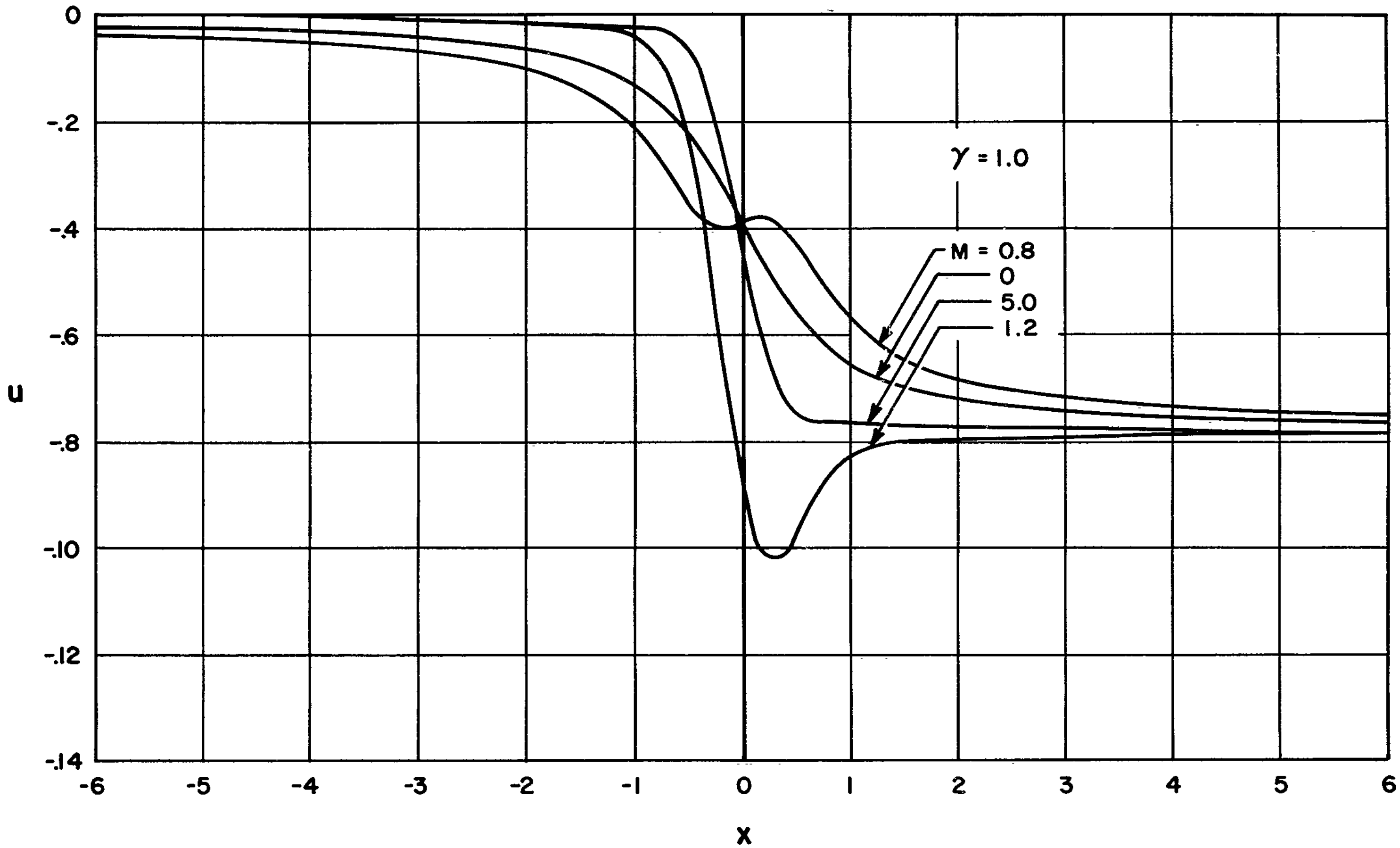

Fig. 13a This is the tangential velocity at the wall when the flow is impeded by the magnetic field due to a linear dipole at unit distance below the wall. $\gamma = 1.0$.

7212

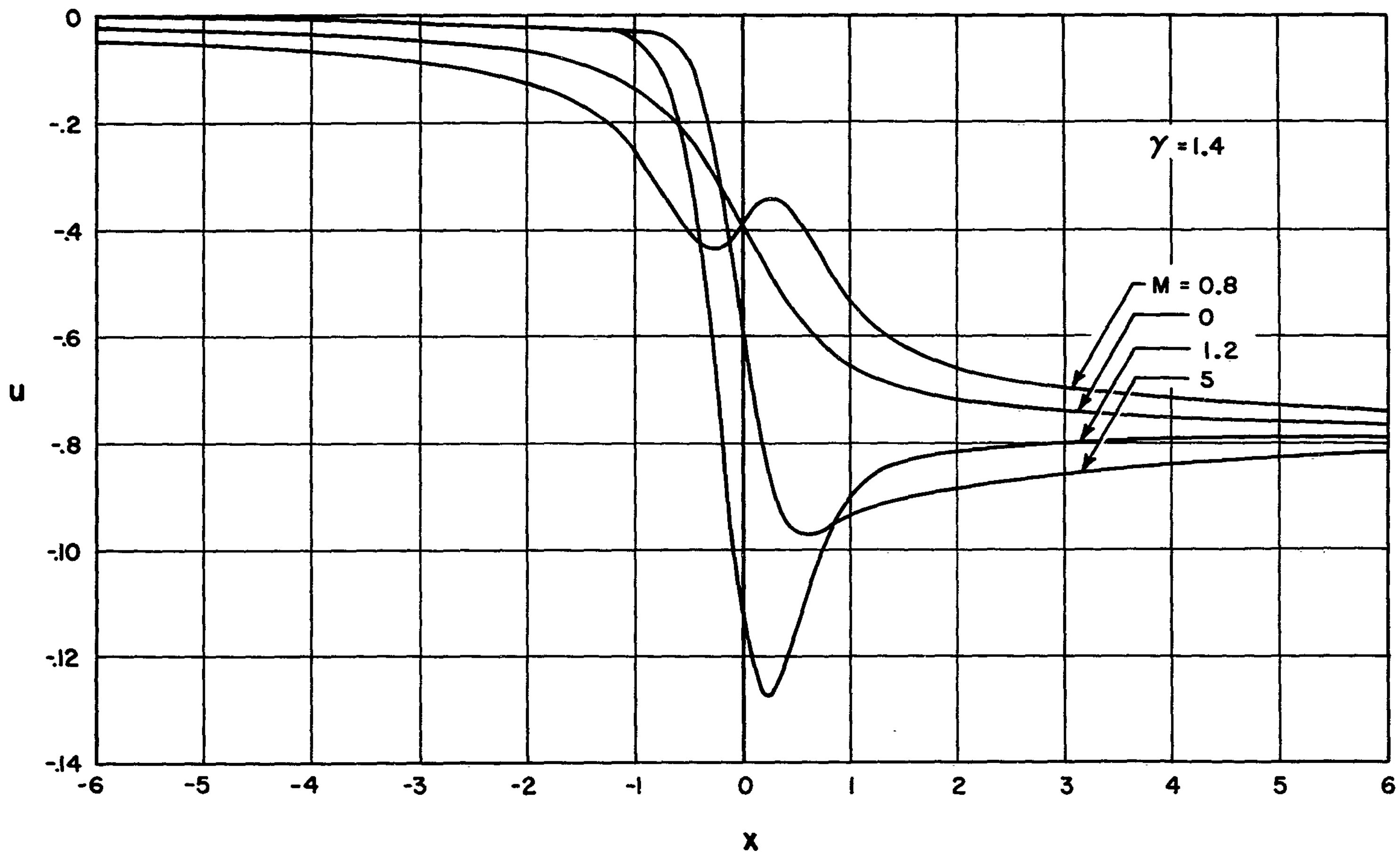

Fig. 13b This is the tangential velocity at the wall when the flow is impeded by the magnetic field due to a linear dipole at unit distance below the wall. $\gamma = 1.4$.

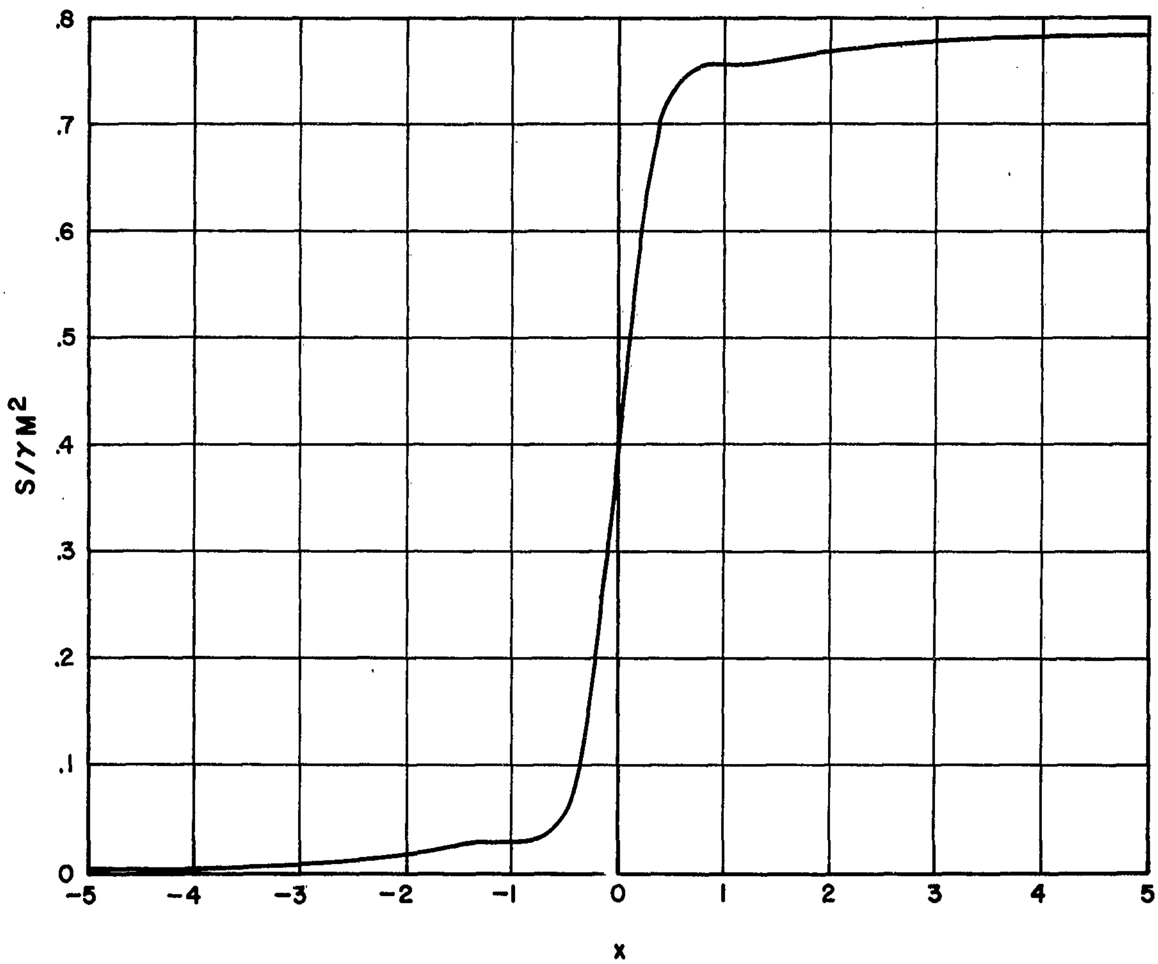

Fig. 14 This is the entropy along a wall when the flow is impeded by the magnetic field due to a linear dipole.

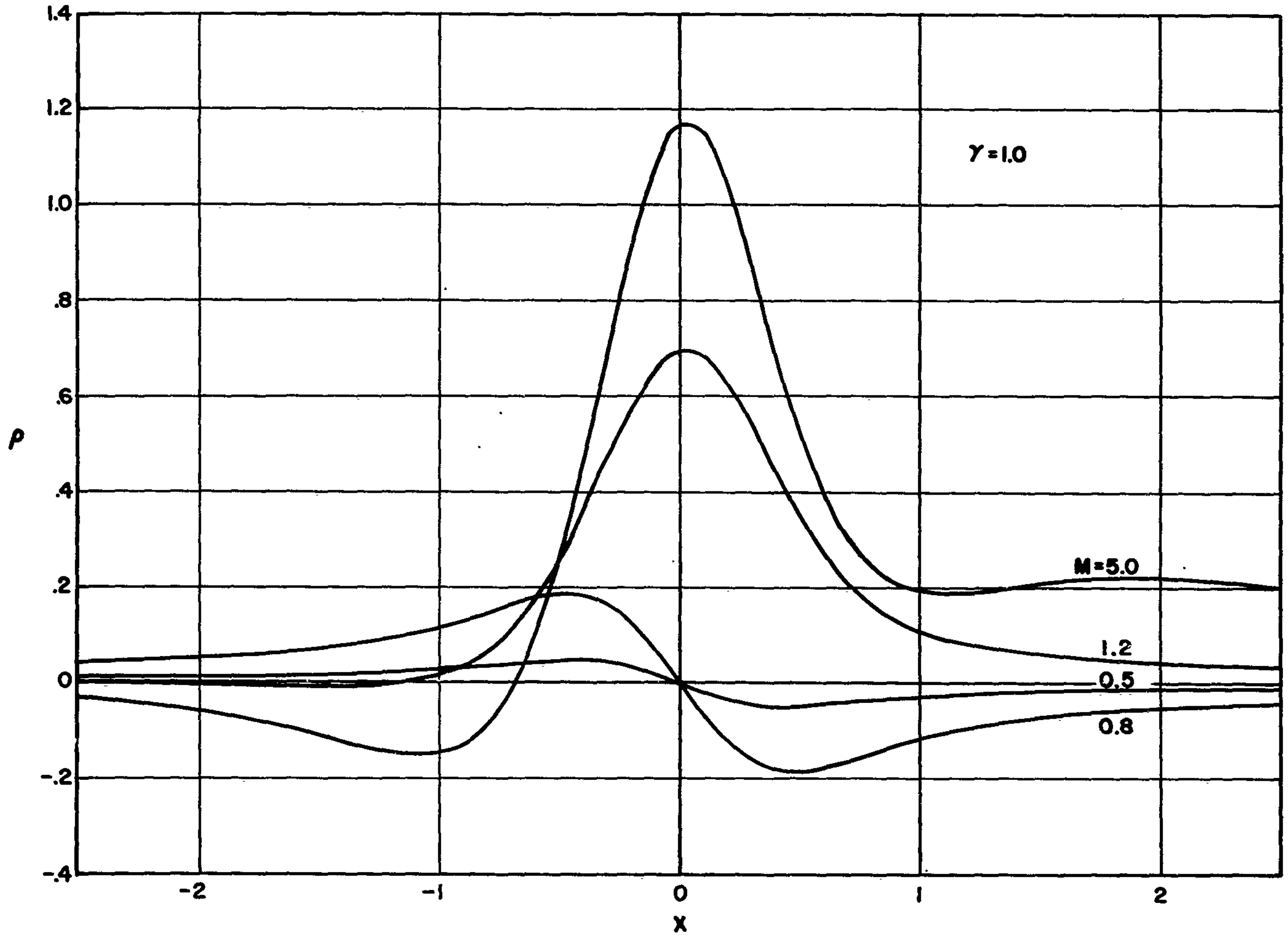

Fig. 15a This is the density on a wall when the flow is impeded by the magnetic field due to a linear dipole at unit distance below the wall. $\gamma = 1.0$.

7215

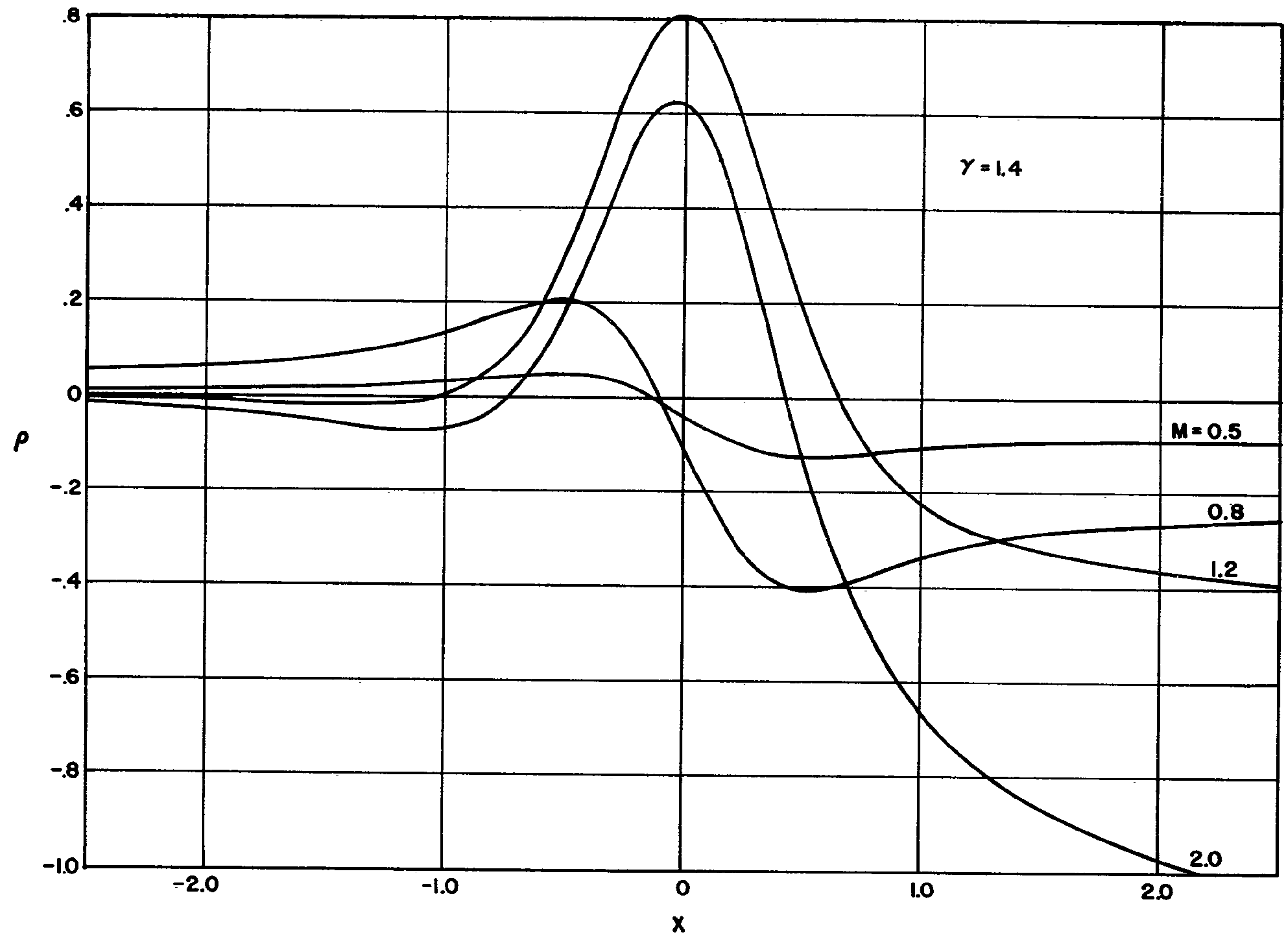

Fig. 15b This is the density on a wall when the flow is impeded by the magnetic field due to a linear dipole at unit distance below the wall. $\gamma = 1.4$.

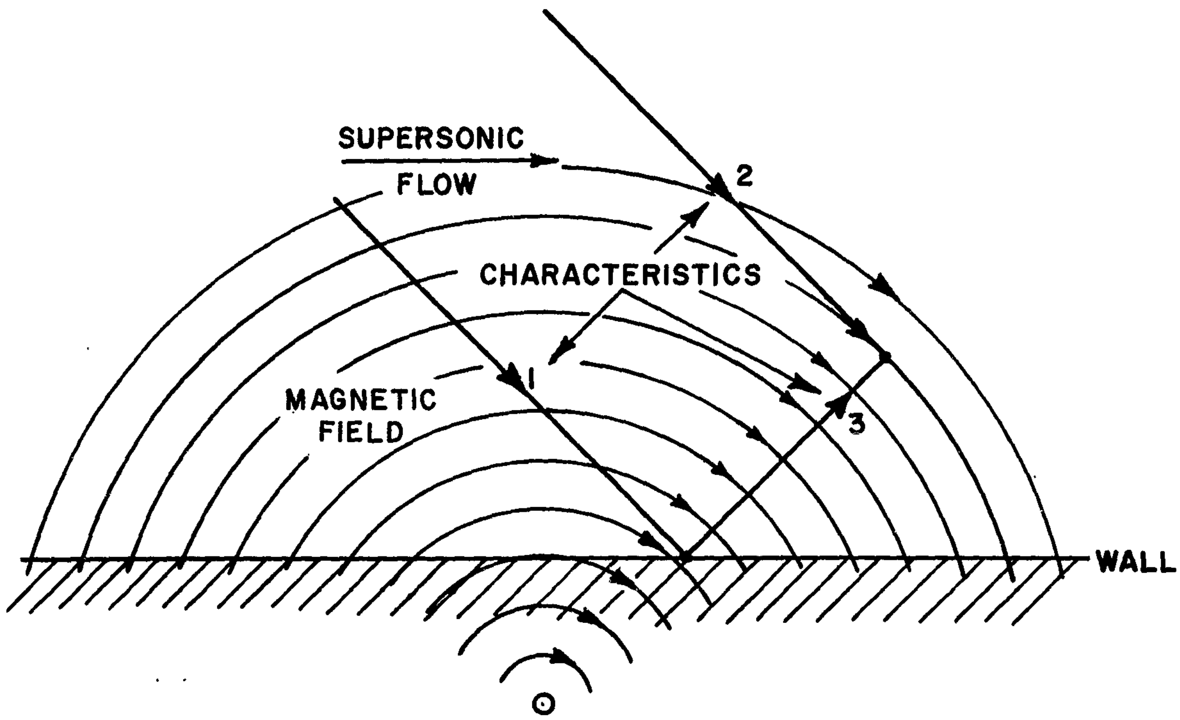

Fig. 16 This is a sketch showing the characteristics used in calculating the perturbations to a supersonic flow.

7217

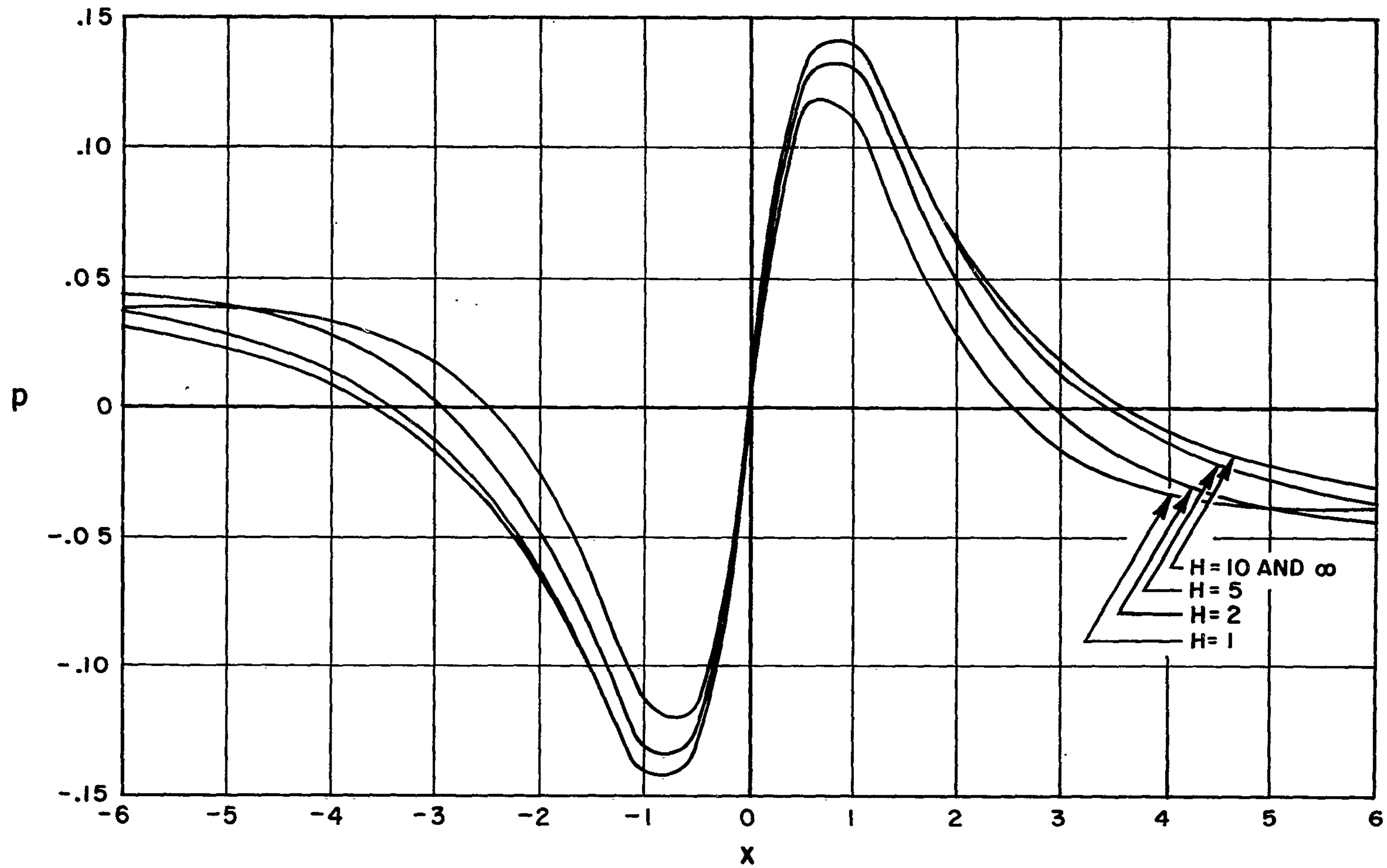

Fig. 17 This is the pressure on a wall when the flow is impeded by the magnetic field due to a current flowing in a single wire at unit distance below the wall and the flow is conducting between the wall and the line y = H.

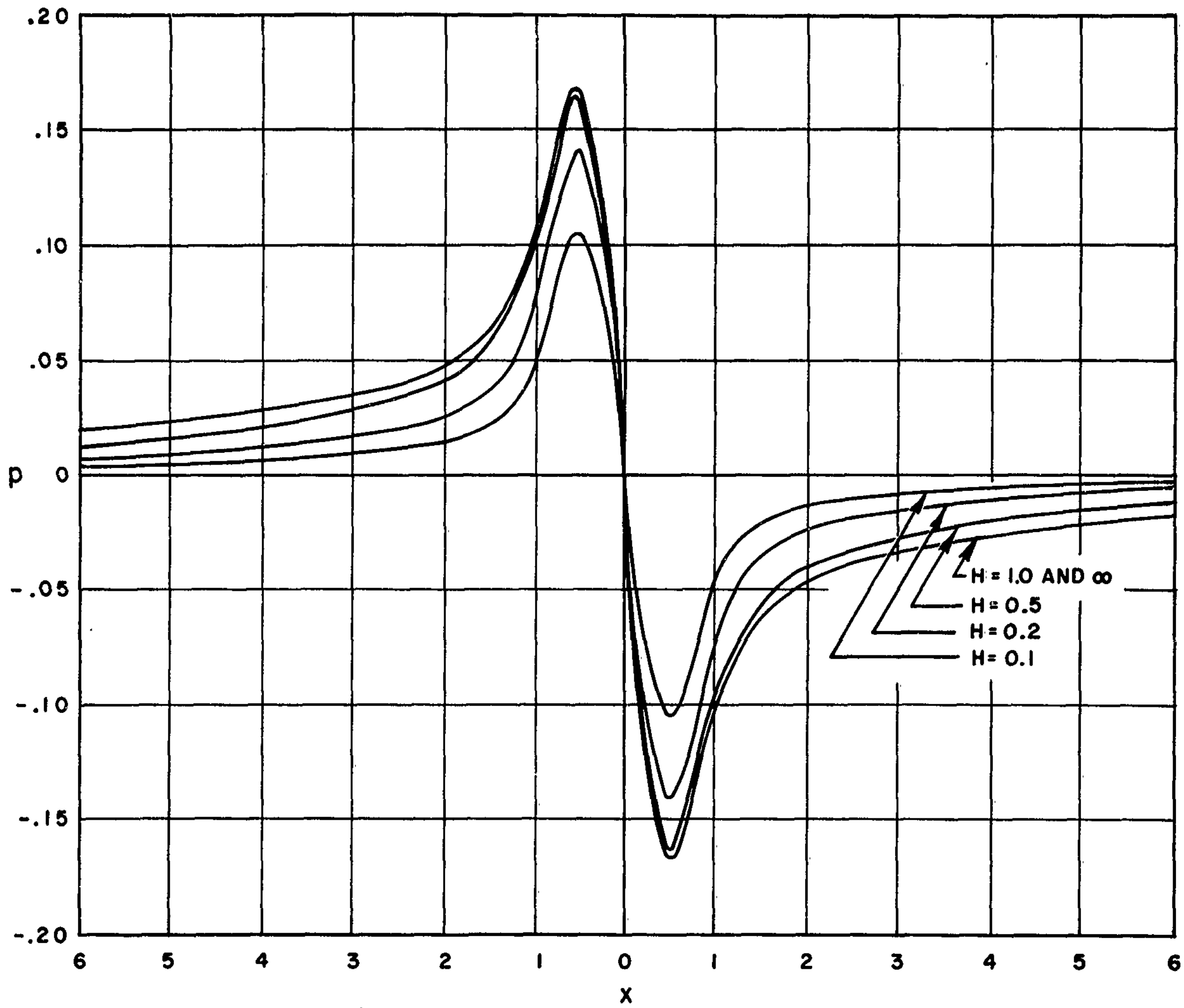

Fig. 18 This is the pressure on a wall when the flow is impeded by the magnetic field due to a linear dipole at unit distance below the wall and the fluid is conducting between the wall and the line y = H.

APPENDIX

Note Added in Proof

Since this paper was compiled, two additional papers on this general subject have come to the author's attention; these are the works of Ehlers[6] and Morioka.[7]

Ehlers treats the same physical problem as is considered here, but considers axially symmetric geometries and obtains solutions by integral transform methods. The results appear in a form which does not lend itself to simple interpretation. It is interesting to note that particular solutions to the axially symmetric incompressible problem analogous to those given here for two-dimensional flow may be obtained. In fact, the equations governing the axi-symmetric case are:

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial r} = b_x b_r \; ; \; \frac{1}{r} \frac{\partial}{\partial r}(r v) - \frac{\partial p}{\partial x} = b_r^2$$

where r is the radial coordinate. The magnetic field components satisfy:

$$\frac{\partial b_r}{\partial x} - \frac{\partial b_x}{\partial r} = 0 \; ; \; \frac{\partial b_x}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r}(r b_r) = 0$$

A particular solution is then

$$v_p = \frac{1}{2} b_r \int_{-\infty}^{x} b_x dx + \int_{-\infty}^{x} b_x b_r dx \tag{A1}$$

$$p_p = -\frac{1}{2} b_x \int_{-\infty}^{x} b_x dx \quad . \tag{A2}$$

The particular example treated by Ehlers involves the magnetic field due to a circular coil. The field components in this case do not permit the evaluation of the integrals (A1) and (A2) in closed form, but it seems clear that simple axisymmetric fields could be treated in detail using the particular integrals given above.

Morioka, on the other hand, treats precisely the two-dimensional problem treated here, with the magnetic field being due to a linear dipole. His method is somewhat similar to that of the present paper, but the results disagree; certain misprints have been found in Morioka's paper, but it is not known whether all the differences in results can be attributed to misprints.

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Avco-Everett Research Laboratory, Everett, Massachusetts
EXACT SOLUTIONS TO A CLASS OF TWO-DIMENSIONAL MAGNETOHYDRODYNAMIC FLOW PROBLEMS AT LOW CONDUCTIVITY, by Richard H. Levy. November 1961. 59 p. incl. illus. (Avco-Everett AMP 70; AFBSD-TN-61-31) (Contract AF 04(694)-33)

Unclassified report

Exact solutions are presented to a number of small perturbation magnetohydrodynamic flow problems. The conditions under which the solutions are obtained are as follows:
1. The flow is two-dimensional, and is only slightly perturbed from a uniform flow. 2. The magnetic field vector is also two-dimensional and lies in the plane of the flow. 3. The distortion of the applied field by the induced currents is neglected. 4. Physical boundaries on the flow are one or two infinite plates parallel to the flow direction. 5. The conductivity of the fluid is a scalar quantity, but may vary with position. With these assumptions, the perturbations to the flow are calculated for various magnetic fields (chiefly those due to a current flowing in a single wire, and a linear dipole)

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2. Flow, Two-Dimensional.
3. Magnetic Fields.
I. Title.
II. Levy, Richard H.
III. Avco-Everett AMP 70.
IV. AFBSD-TN-61-31.
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